Rotational Kinematics

We discussed the basics of rotational kinematics in Chapter 5; we complete the story here, including the kinematics equations for constant angular acceleration.

It's probably a good idea to review Chapter 5 before you begin this chapter. Recall that the basic quantities of rotational kinematics are angular position, angular displacement, angular velocity, and angular acceleration; period and frequency.

Recall the basic equation connecting period and frequency, developed in Chapter 5 for uniform circular motion:

\[ T = \frac{1}{f} \]

Recall the definition of radian measure:

\[ \theta = \frac{s}{r} \]

This is equivalent to

\[ s = r \theta \]
Recall the basic connections between linear kinematical quantities and angular kinematical quantities, which follow from the previous relation:

\[ \mathbf{v}_T = r \omega \]

\[ \mathbf{a}_T = r \alpha \]

Recall the distinction between tangential acceleration and centripetal acceleration for an object moving in a circle: centripetal acceleration points towards the centre of the circle and serves to change the direction of the velocity vector, whereas the tangential acceleration is directed along the tangent to the circle at the location of the moving object, and serves to change the speed of the object. The angular acceleration serves to change the angular velocity.

The following diagrams illustrate these points:

\[ a_c = \frac{v^2}{r} \]  

\[ \alpha \rightarrow \text{increases or decreases speed } |\mathbf{v}| \]

\[ \alpha \rightarrow \text{increases or decreases } \omega \]
In the previous pair of diagrams, note that for uniform circular motion (left diagram) the acceleration vector points towards the centre of the circle; in other words, there is non-zero centripetal component of acceleration, but the tangential component of acceleration is zero. For non-uniform circular motion (right diagram), both the centripetal and tangential components of acceleration are non-zero, so the net acceleration is not towards the centre of the circle.

Similar comments apply to motion that is along curved paths that are not necessarily circles:

\[ \alpha \rightarrow \text{changes direction but NOT speed} \]

\[ \alpha_T \rightarrow \text{changes speed but NOT direction} \]
If the angular acceleration is constant, then we can write kinematics equations for angular variables that are analogous to kinematics equations for motion in a straight line:

\[ \omega = \frac{\Delta \theta}{\Delta t} \]

**If \( \omega \) is constant, then**

If \( \omega \) is not constant, then

\[ \omega = \frac{d\theta}{dt} \]

**If \( \alpha \) is constant, then**

\[ \alpha = \frac{\Delta \omega}{\Delta t} \]

If \( \alpha \) is not constant, then

\[ \alpha = \frac{d\omega}{dt} \]

If the angular acceleration is constant, then we can write kinematics equations for angular variables that are analogous to kinematics equations for motion in a straight line:

Linear motion \((\alpha = \text{constant})\)

\[ v = v_0 + \alpha t \]

Circular motion \((\alpha = \text{constant})\)

\[ \omega = \omega_0 + \alpha t \]
Example: The Moon is $3.85 \times 10^8$ m from Earth and has a diameter of $3.48 \times 10^6$ m. How big an object would you have to hold at arm's length (assumed to be 70 cm) to completely cover the Moon?

Solution: Calculate the angular size of the Moon as seen from Earth and then use the same angular size to determine the size of the object that could "cover" the Moon:

$$
\theta = \frac{s}{r}
$$

$$
\theta = \frac{3.48 \times 10^6 \text{ m}}{3.85 \times 10^8 \text{ m}}
$$

$$
\theta = 0.00904 \text{ rad}
$$

For the "covering" object,

$$
s = r \theta
$$

$$
s = (70 \text{ cm})(0.00904 \text{ rad})
$$

$$
s = 0.63 \text{ cm}
$$
Interestingly, the Sun's angular diameter and the Moon's angular diameter, as observed from the surface of the Earth, are almost exactly the same, which makes total eclipses of the Sun possible.

Example: The Sun orbits the centre of the Milky Way galaxy. The radius of the circular orbit is about $2.2 \times 10^{20}$ m and the Sun's angular speed in its orbit is about $8.25 \times 10^{-16}$ rad/s. How many years does it take for the Sun to make one revolution about the centre of the Milky Way?

Solution: Is it clear that the radius of the orbit is irrelevant? If not, think about this again once you've worked through the solution. You might like to think about a merry-go-round; the horses at different radii may be moving at different speeds, but they move at the same angular speed, so they all take the same amount of time to go around once.

(This is not to imply that the galaxy is a rigid object so that all stars orbit at the same angular speed; no, this is not so. The point of bringing up the merry-go-round is to emphasize the fact that the connection between angular speed and period is independent of radius.)

$$\omega = 8.25 \times 10^{-16} \text{ rad/s}$$

$$\frac{1}{\omega} = \frac{1}{8.25 \times 10^{-16}} \frac{s}{\text{rad}}$$

$$T = \frac{2\pi}{8.25 \times 10^{-16}} \frac{s}{(2\pi \text{ rad})}$$
Study the solution carefully and then note that a short-cut is to use the formulas

\[ \omega = 2\pi f \quad \text{and} \quad f = \frac{1}{T} \]

to conclude that

\[ T = \frac{2\pi}{\omega} \quad \text{rad/s} \]

Example: A disk of radius 5.0 cm starts from rest and rotates with a constant angular acceleration of 0.6 rad/s². After 4.3 s, determine
(a) the speed of a point on the edge of the disk.
(b) the centripetal and tangential components of acceleration.

Solution: (a)

\[ \Delta \omega = \alpha \Delta t \rightarrow \omega - \omega_i = \left(\frac{0.6 \text{ rad}}{s^2}\right)(4.3 \text{ s}) \]

\[ \omega - 0 = 2.58 \text{ rad/s} \]

\[ v = \omega r = \left(2.58 \frac{\text{ rad}}{s}\right)(0.05 \text{ m}) \]
Example: The device in the figure is used to determine the speed of a bullet. The two disks rotate with an angular speed of 95.0 rad/s and are separated by a distance of 0.850 m. The bullet passes through the disks, and the angular displacement between the two bullet holes is 0.240 rad. Determine the speed of the bullet.

Solution: The time needed for the apparatus to rotate through 0.240 rad is the same as the time needed for the bullet to move between the two disks. Let's write an equation setting these two times equal to each other, and solve the equation for the speed of the bullet:

\[ \omega = \frac{\Delta \theta}{\Delta t} \rightarrow \Delta t = \frac{\Delta \theta}{\omega} \]
\[ \omega = \frac{\Delta \theta}{\Delta t} \Rightarrow \Delta t = \frac{\Delta \theta}{\omega} \]

\[ d = v \Delta t \Rightarrow \Delta t = \frac{d}{v} \]

\[ \therefore \quad \frac{\Delta \theta}{\omega} = \frac{d}{v} \]

\[ v \Delta \theta = d \omega \]

\[ v = \frac{d \omega}{\Delta \theta} \]

\[ v = \frac{(0.85 \text{ m})(95.0 \text{ rad/s})}{0.240 \text{ rad}} \]

\[ v = 336 \text{ m/s} \]

If one needs to inject a stream of particles into an experiment, all of which have the same velocity, then one way is to pass them through a similar apparatus that has the holes pre-drilled. Particles with just the right velocity pass through both holes and into the experimental chamber, whereas particles that don't have the right velocity bounce off the solid parts of the disks and don't make it through the holes into the experimental chamber. Such an apparatus is called a **velocity selector**.

Example: A figure skater spinning with an angular velocity of 15 rad/s comes to a stop while spinning through an angular displacement of 5.1 rad. Determine (a) her average angular acceleration and (b) the time during which she comes to rest.
Solution: Assuming that the angular acceleration is constant, we can use the kinematics equations for constant angular acceleration.

\[ \Delta \theta = 5.1 \text{ rad} \]

\[ \omega_0 = 15 \text{ rad/s} \]

\[ \omega = 0 \]

\[ \alpha = ? \]

\[ \Delta t = ? \]

(a) \[ \omega^2 - \omega_0^2 = 2\alpha \Delta \theta \]

\[ \alpha = \frac{\omega^2 - \omega_0^2}{2 \Delta \theta} \]

\[ \alpha = \frac{0^2 - (15 \text{ rad/s})^2}{2 (5.1 \text{ rad})} \]

\[ \alpha = -22.1 \text{ rad/s}^2 \]

(b) \[ \Delta \theta = \omega_0 \Delta t + \frac{1}{2} \alpha (\Delta t)^2 \]

\[ 5.1 = 15 \Delta t + \frac{1}{2} (-22.1)(\Delta t)^2 \]

\[ 11.03(\Delta t)^2 - 15 \Delta t + 5.1 = 0 \]

\[ \Delta t = \frac{15 \pm \sqrt{15^2 - 4(11.03)(5.1)}}{2(11.03)} \]
Example:

A dentist causes the bit of a high-speed drill to accelerate from an angular speed of \(1.05 \times 10^4\) rad/s to an angular speed of \(3.14 \times 10^4\) rad/s. In the process, the bit turns through \(1.88 \times 10^4\) rad. Assuming a constant angular acceleration, how long would it take the bit to reach its maximum speed of \(7.85 \times 10^4\) rad/s, starting from rest?

Solution: First determine the magnitude of the angular acceleration during the first process, then use it to determine the time in the second process, with the understanding that the angular acceleration is the same in both cases.

\[
\Delta t = \frac{15 \pm \sqrt{0}}{22.06}
\]

\[
\Delta t = 0.68 \text{ s}
\]

An alternative solution for Part (b) is:

\[
\Delta \theta = \frac{1}{2} (w_0 + w) \Delta t
\]

\[
\Delta t = \frac{2\Delta \theta}{w_0 + w}
\]

\[
\Delta t = \frac{2(5.1)}{0 + 15}
\]

\[
\Delta t = 0.68 \text{ s}
\]

Which solution for Part (b) do you prefer?
For the first process:

\[ \omega^2 - \omega_0^2 = 2 \alpha \Delta \theta \]

\[ \alpha = \frac{\omega^2 - \omega_0^2}{2 \Delta \theta} = \frac{(3.14 \times 10^4)^2 - (1.05 \times 10^4)^2}{2 (1.88 \times 10^4)} \]

\[ \alpha = 2.329 \times 10^4 \text{ rad/s}^2 \]

For the second process:

\[ \alpha = \frac{\Delta \omega}{\Delta t} \rightarrow \Delta t = \frac{\Delta \omega}{\alpha} \]

\[ \Delta t = \frac{\omega - \omega_0}{\alpha} = \frac{7.85 \times 10^4}{2.329 \times 10^4} \]

\[ \Delta t = 3.37 \text{ s} \]

So the drill comes up to speed in 3.37 s if it starts from rest.

Example: An auto race takes place on a circular track. A car completes one lap in a time of 18.9 s, with an average tangential speed of 42.6 m/s. Determine

(a) the average angular speed and

(b) the radius of the track.
Solution: (a) My first thought is to use the relation
\[ v_T = \omega r \rightarrow \omega = \frac{v_T}{r} \]

But I'm not too happy about this, because I don't know the radius of the track. OK, let's think again. We know the period, which is 18.9 s, and we know that the frequency is the reciprocal of the period, so we can calculate the frequency. Isn't the angular speed more-or-less the frequency? Yes. OK, so there must be a simple relation between the angular speed and the period. (Remember, every part of a merry-go-round has the same angular speed, independent of the distance of each part from the centre.)

\[ \omega = 2\pi f = 2\pi \left( \frac{1}{T} \right) = \frac{2\pi}{T} \]

\[ \therefore \omega = \frac{2\pi \text{ rad}}{18.9 \text{ s}} = 0.33 \text{ rad/s} \]

(b) Now that we know the angular speed, we can go back to the first relation we wrote down:

\[ v_T = \omega r \rightarrow r = \frac{v_T}{\omega} \]

\[ r = \frac{4 \cdot 2.6 \text{ m/s}}{0.33 \text{ rad/s}} \]

\[ r = 12.8 \text{ m} \]

Alternative solution for Part (b):

\[ s = v_T T \]

\[ 2\pi r = v_T T \]
Example: The earth has a radius of \( 6.38 \times 10^6 \) m and turns on its axis once every 23.9 h.

(a) What is the tangential speed (in m/s) of a person living in Ecuador, a country that lies on the equator?

(b) At what latitude (i.e., the angle \( \theta \) in the drawing) is the tangential speed one-third that of a person living in Ecuador?

Solution: Like a merry-go-round, every point on the Earth has
the same angular speed, so it's probably a good idea to calculate it as a start. Indeed, once we have the angular speed, we'll be able to solve Part (a) in short order.

\[ \omega = \frac{2\pi}{T} \]

\[ \omega = \frac{2\pi}{23.9 \text{ h}} \]

\[ \omega = \frac{2\pi}{(23.9)(3600)} \text{ rad/s} \]

\[ \omega = 7.3026 \times 10^{-5} \text{ rad/s} \]

(a) Thus, the tangential speed of a point on the equator is

\[ v_T = r\omega \]

\[ v_T = (6.38 \times 10^6)(7.3026 \times 10^{-5}) \]

\[ v_T = 466 \text{ m/s} \]

(b) The tangential speed is proportional to the radius:

\[ v_T = r\omega \]

where the angular speed is the proportionality constant. Thus, the latitude at which the tangential speed is one-third of the value at the equator has one-third the radius of the equator. Thus,
Having solved the problem, isn't there one thing about this problem that bothers you? I thought there were 24 hours in a day, so why do they say that it takes 23.9 hours for the Earth to rotate one complete rotation?

It's worth giving some thought to this; yes, there is indeed a difference between the length of a day and the time it takes the Earth to rotate once on its axis! The difference is about 4 minutes. Why? You might like to draw some diagrams of Earth rotating on its axis as it revolves in its orbit around the Sun to figure this out.

Once you've given this some thought, if you would like to learn more about this, search "solar day and sidereal day" online or in a good introductory astronomy textbook. One good online explanation (with a good diagram) is here:

http://community.dur.ac.uk/john.lucey/users/e2_solsid.html
Example: A racing car travels with a constant tangential speed of 75.0 m/s around a circular track of radius 625 m. Determine (a) the magnitude of the car's total acceleration, and (b) the direction of its total acceleration relative to the radial direction.

Solution: Because the car's speed around the track is constant, there is no tangential component of acceleration. However, there is a centripetal component of acceleration; otherwise, the car would move in a straight line, not a circle. Calculating the centripetal component of acceleration will be enough to determine the total acceleration.

(a) \[ a_c = \frac{v_t^2}{r} \]

\[ a_c = \frac{75^2}{625} \]

\[ a_c = 9 \text{ m/s}^2 \]

Thus, the magnitude of the acceleration is 9 m/s².

(b) The acceleration is directed towards the centre of the circle. Another way of saying this is that the acceleration is directed radially inward.

Example: An electric drill starts from rest and rotates with a constant angular acceleration. After the drill has rotated
through a certain angle, the magnitude of the centripetal acceleration of a point on the drill is twice the magnitude of the tangential acceleration. Determine the angle through which the drill rotates by this point.

Solution: I have no idea how to solve this problem. So what shall I do, seeing how I have no idea how to solve this? Well, a good strategy is to translate the condition given in the problem into mathematical language, and then examine what we have to see what else can be done from there.

OK, so the condition involves the centripetal acceleration and the tangential acceleration, so it's good to start by writing expressions for each:

\[ a_c = \frac{v^2}{r} \quad a_T = r \alpha \]

Ultimately we have to determine an angle, so perhaps it's wise to convert the linear velocity to angular velocity in the expression for the centripetal acceleration:

\[ v = r \omega \]

\[ \quad \rightarrow \quad a_c = \frac{(r \omega)^2}{r} = \frac{r^2 \omega^2}{r} \]

\[ a_c = r \omega^2 \]

I'm still not sure how to solve the problem, but now that we have expressions for the centripetal and tangential acceleration, we can write an equation that expresses the condition given: The magnitude of the centripetal acceleration is twice the magnitude of the tangential acceleration.

\[ a_c = 2 a_T \]
That is,

\[ r \omega^2 = 2r \alpha \]

which is equivalent to

\[ \omega^2 = 2 \alpha \]

OK, so we've got a relation between the angular speed and the angular acceleration. But we need to determine the angular displacement. Is there a kinematics equation that relates angle with angular speed or angular acceleration? Why yes, there is. Choose the one that does not include time, because we don't know anything about how long this process takes, and we're not asked about the time. Thus, choose

\[ \omega^2 - \omega_0^2 = 2 \alpha \Delta \theta \]

Now use the facts that \( \omega_0 = 0 \)
and \( \omega^2 = 2 \alpha \) to obtain

\[ 2 \alpha - 0 = 2 \alpha \Delta \theta \]
\[ 2 \alpha = 2 \alpha \Delta \theta \]
\[ \Delta \theta = 1 \text{ rad.} \]

Rolling motion

If a wheel rolls without slipping, there is a relationship between the tangential speed of the edge of the wheel and the forward speed of the centre of the wheel. Consider the following figure, which
depicts one of the wheels of a car moving to the right:

The forward distance travelled by the car, $d$, is equal to the arc, $s$, through which the tire turns, where

$$s = r \theta$$

Thus, if the car is moving at a constant speed, the car's speed $v$ is connected to the angular speed of the wheels by

$$\frac{s}{\Delta t} = r \frac{\theta}{\Delta t}$$

$$v = r \omega$$

If the car is accelerating, then the acceleration of the car is the same as the tangential acceleration of a point on the edge of the wheel, so

$$\frac{v}{\Delta t} = r \frac{\omega}{\Delta t}$$

$$a_T = r \alpha$$

Thus, the same relations that hold for a rotating object also hold for a wheel that rolls without slipping, with the additional point
that the acceleration of the centre of the rolling wheel is the same as the tangential acceleration of a point on the edge of the wheel.

Example: An automobile tire has a radius of 0.330 m, and its centre moves forward with a linear speed of 15.0 m/s. 
(a) Determine the angular speed of the wheel. 
(b) Relative to the axle, what is the tangential speed of a point located 0.175 m from the axle?

Solution:

\[
\begin{align*}
(a) \quad \nu &= r \omega \\
\omega &= \frac{\nu}{r} \\
\omega &= \frac{15 \text{ m/s}}{0.33 \text{ m}} \\
\omega &= 45 \text{ rad/s}
\end{align*}
\]

(b) Every point on the wheel has the same angular speed. Thus,

\[
\begin{align*}
\nu_T &= r \omega \\
\nu_T &= (0.175 \text{ m}) \left( \frac{15}{0.33} \right) \\
\nu_T &= 7.95 \text{ m/s}
\end{align*}
\]

Example: Suppose you are riding a stationary exercise bicycle, and the electronic meter indicates that the wheel is rotating at 9.1 rad/s. The wheel has a radius of 0.45 m. If you ride the bike for 35 min, how far would you have gone if the bike could
Example: A dragster starts from rest and accelerates down a track. Each tire has a radius of 0.320 m and rolls without slipping. At a distance of 384 m, the angular speed of the wheels is 288 rad/s. Determine
(a) the linear speed of the dragster, and
(b) the magnitude of the angular acceleration of its wheels.

Solution:

(a) \[ v = r \omega \]
\[ v = (0.320 \text{ m}) (288 \text{ rad/s}) \]
\( v = 92 \text{ m/s} \)

(b) When the car has moved 384 m, its wheels have turned through an angle \( \theta \), where

\[
\theta = \frac{s}{r}
\]

\[
\theta = \frac{384 \text{ m}}{0.320 \text{ m}}
\]

\[
\theta = 1200 \text{ rad}
\]

Now use a kinematics equation to determine the angular acceleration:

\[
\omega^2 - \omega_0^2 = 2 \alpha \theta
\]

\[
\alpha = \frac{\omega^2 - \omega_0^2}{2 \theta}
\]

\[
\alpha = \frac{288^2 - 0}{2(1200)}
\]

\[
\alpha = 34.6 \text{ rad/s}^2
\]