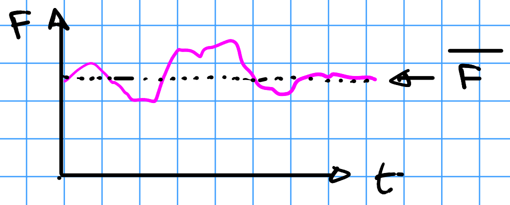
Linear momentum, $\vec{p}$



$\vec{F} \neq$ const $\Rightarrow$ $\vec{a} \neq$ const

$\Rightarrow$ average $\bar{F}$ $\Rightarrow$ average $\bar{a} = \dfrac{\bar{F}}{m}$

include the direction: $\bar{\vec{a}} = \dfrac{\bar{\vec{F}}}{m}$

By definition: $\bar{\vec{a}} = \dfrac{\vec{v} - \vec{v}_0}{\Delta t} = \dfrac{\Delta \vec{v}}{\Delta t}$

re-write N2L: $\bar{\vec{F}} = m\bar{\vec{a}} = m\dfrac{\Delta \vec{v}}{\Delta t} = \dfrac{\Delta(m\vec{v})}{\Delta t} = \dfrac{m\vec{v} - m\vec{v}_0}{\Delta t}$

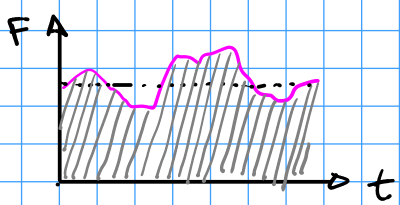
define $\boxed{\vec{p} \equiv m\vec{v}}$ = linear momentum, $\vec{p}$ is a vector

$[p] = \dfrac{kg\,m}{s}$

$\Rightarrow$ $\bar{\vec{F}} = \dfrac{\Delta \vec{p}}{\Delta t}$ or $\bar{\vec{F}}\Delta t = \Delta \vec{p}$ = impulse

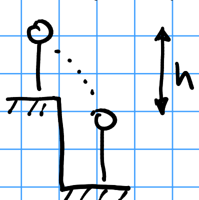
i.e. N2L: "force = rate of change of momentum".

- visualize: total impulse = area under F(t) curve



N2L: $\boxed{\text{impulse} = \text{change in momentum}}$

Ex. jumping without breaking ankles



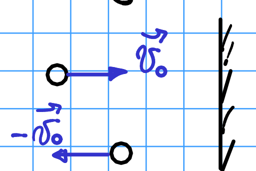
$\cancel{m}gh = \dfrac{\cancel{m}v^2}{2}$ $\quad v = \sqrt{2gh}$

impact: $\left.\begin{matrix} v_0 = v \\ v_f = \emptyset \end{matrix}\right\}$ $\bar{F} = \dfrac{\Delta P}{\Delta t} = \dfrac{mv}{\Delta t_{landing}} \propto \dfrac{1}{\Delta t_{landing}}$

$\Rightarrow$ bend knees!

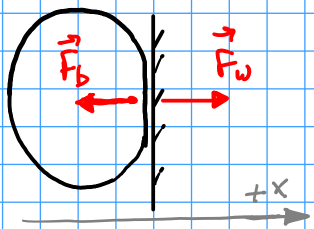
Also: air bags in cars, "rolling with the punches" are all about extending Δt, and thus making $\bar{F}$ smaller

Ex. a tennis ball



Elastic: $v_f = -V_0$

$\Delta p = mV_f - mV_0 = m(-V_0) - mV_0 = -2mV_0$



$$F_{\text{on ball}} = \frac{\Delta p}{\Delta t_{\text{collision}}} = \frac{-2mV_0}{\Delta t}$$

$$\text{N3L: } F_{\text{on wall}} = -F_{\text{on ball}} = \frac{2mV_0}{\Delta t} > \emptyset$$

$\Rightarrow$ $\exists$ force on the wall due to the collision.

Ex a tennis ball, 55.0g, $V_0 = 22.0\,\frac{m}{s}$, once/sec.

$$\Delta p = mV_f - mV_0 = -2mV_0 = -2 \times 0.055\,kg \times 22.0\,\frac{m}{s} = -2.42\,\frac{kg\,m}{s}$$

collision takes $\Delta t = 1.25\,ms$

$$\overline{F}_{ball} = \frac{\Delta P}{\Delta t} = \frac{-2.42}{1.25 \times 10^{-3}} \left(\frac{kg\,m}{s \cdot s} = N\right) = -1.94 \times 10^{3}\,N$$

"to the left"

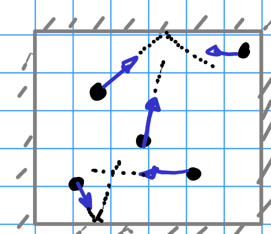
Inelastic : some kinetic energy is lost in the collision.

Suppose $V_f = -19.0\,\frac{m}{s}$

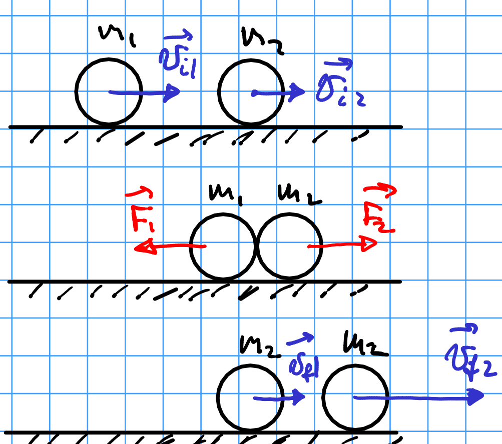
$$\Rightarrow \Delta p = mV_f - mV_0 = -2.26\,\frac{kg\,m}{s}, \quad \overline{F}_{ball} = \frac{-2.255\ldots}{1.25 \cdot 10^{-3}} = -1.80 \times 10^{3}\,N$$

and, of course, $\overline{F}_{wall} = -\overline{F}_{ball} = 1.80 \times 10^{3}\,N$

This type of force is responsible for the pressure of a gas confined to a container:



- internal and external forces in a collision



← before

collision takes Δt

N3L: $\vec{F}_1 = -\vec{F}_2$

← after

Forces on:

	external	internal
m_1 :	$\vec{W}_1, \vec{F}_{N1}$	$\vec{F}_1$
m_2 :	$\vec{W}_2, \vec{F}_{N2}$	$\vec{F}_2$

relative to the system of the two balls, these are:

$$\Delta\vec{p}_1 = \left(\sum \vec{F}_i\right)_1 \Delta t = \left[\left(\sum \vec{F}_{ext}\right)_1 + \vec{F}_1\right]\Delta t$$

$$\Delta\vec{p}_2 = \left(\sum \vec{F}_i\right)_2 \Delta t = \left[\left(\sum \vec{F}_{ext}\right)_2 + \vec{F}_2\right]\Delta t$$

$$\Delta\vec{p}_{total} = \left[\left(\sum F_{ext}\right)_{1\&2} + \vec{F}_1 + \vec{F}_2\right]\Delta t \quad , \text{ but } \vec{F}_2 = -\vec{F}_1$$

$$\Rightarrow \quad \Delta\vec{p}_{total} = \left(\sum F_{ext}\right)_{total} \Delta t$$

or $\left(\sum F_{ext}\right)_{total} = \boxed{\vec{F}_{ext,\,total} = \dfrac{\Delta\vec{p}_{total}}{\Delta t}}$

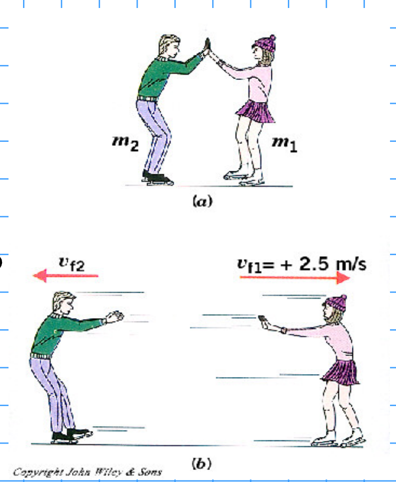
In particular, when $\vec{F}_{ext,\,total/net} = \emptyset \Rightarrow \dfrac{\Delta\vec{p}}{\Delta t} = \emptyset \Rightarrow \vec{p} = \text{const}$

$\boxed{\vec{p} = \text{const}}$ conservation of momentum

for the system as a whole

Note: vectors are involved $\Rightarrow$ shorthand for $\left(\sum F_{ext}\right)_x = \dfrac{\Delta p_x}{\Delta t}$, etc.

Ex. two skaters



"ignore friction"

$\Rightarrow$ no external forces in x-direction

$\Rightarrow p_x = \text{const.} \Rightarrow p_{i,x} = \emptyset \iff p_{f,x} = \emptyset$

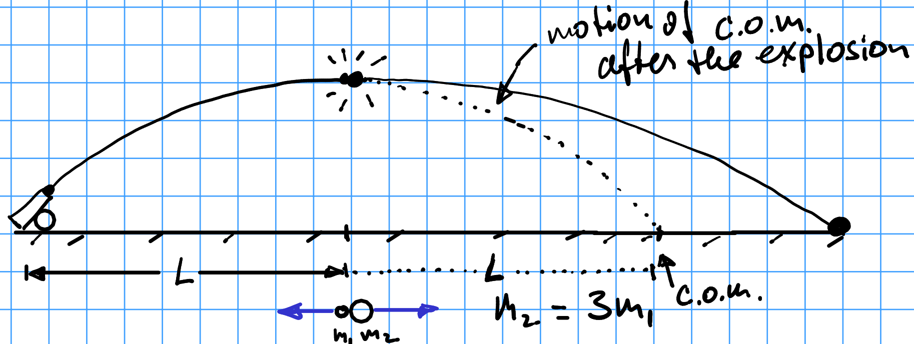
$\Rightarrow m_1 v_{f1} - m_2 v_{f2} = \emptyset$

$$v_{f2} = \frac{m_1 v_{f1}}{m_2} = \frac{54\,\cancel{kg}}{88\,\cancel{kg}} \cdot 2.5\,\frac{m}{s} = 1.5\,\frac{m}{s}$$

answer is +ve, i.e. $\vec{v}_{f2}$ is in the direction shown

Note: c.o.mass of the system of two skaters remains unmoved since $p_{system} = \emptyset = \text{const}$

Ex



The two pieces fly apart horizontally, i.e. $v_y = \emptyset$ for both.

$\Rightarrow$ they land simultaneously

$\Rightarrow$ forces of the explosion are <u>internal</u> to the system of $m_1 + m_2$ $\Rightarrow$ c.o.mass is not affected, it continues on the original trajectory and "lands" @ $R = 2L$

to balance this "see-saw" : $m_1 \cdot 2L = m_2 \cdot x$

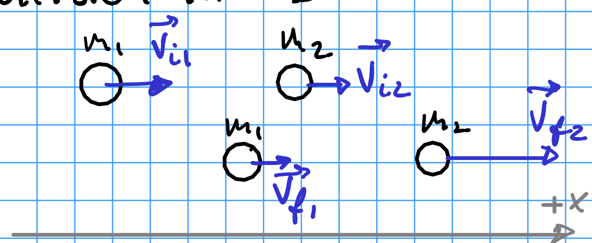
$$\Rightarrow \quad x = 2L \frac{m_1}{m_2} = 2L \frac{\cancel{m_1}}{3\cancel{m_1}} = \frac{2}{3}L$$

$$\Rightarrow \quad 2L + x = 2L + \frac{2}{3}L = \frac{8}{3}L \text{ from origin}$$

Ex a collision in 1D

before:

after:



e.g. billiard balls:
$V_{i2} = \emptyset$, $V_{i1} = V_0$
$\rightarrow$ $V_{f1} = \emptyset$, $V_{f2} = V_0$
i.e. they "exchange" velocities

In general, need 2 equations for 2 unknowns (V_{f1}, V_{f2})

① C.o.K.E: $\frac{1}{2} m_1 V_{i1}^2 + \frac{1}{2} m_2 V_{i2}^2 = \frac{1}{2} m_1 V_{f1}^2 + \frac{1}{2} m_2 V_{f2}^2$

② C.o. Momentum: $m_1 \vec{V}_{i1} + m_2 \vec{V}_{i2} = m_1 \vec{V}_{f1} + m_2 \vec{V}_{f2}$

projecting onto $\xrightarrow{+x}$ $\quad m_1 V_{i1} + m_2 V_{i2} = m_1 V_{f1} + m_2 V_{f2}$

Algebra: re-write CoKE as:

$$m_1 (V_{f1}^2 - V_{i1}^2) = - m_2 (V_{f2}^2 - V_{i2}^2)$$

(*) $\quad m_1 (V_{f1} - V_{i1})(V_{f1} + V_{i1}) = - m_2 (V_{f2} - V_{i2})(V_{f2} + V_{i2})$

Note:
$A^2 - B^2 = (A+B)(A-B)$
Check:
$A^2 - \cancel{AB} + \cancel{BA} - B^2$

Now re-write c.o.M. as:

(**) $\quad m_1 (V_{f1} - V_{i1}) = - m_2 (V_{f2} - V_{i2})$

Divide (*) by (**):

$$V_{f1} + V_{i1} = V_{f2} + V_{i2}$$

$$\Rightarrow \quad V_{f2} = V_{f1} + V_{i1} - V_{i2}$$

Substitute back into C.o.M:

$$\Rightarrow \quad m_1 V_{i1} + m_2 V_{i2} = m_1 V_{f1} + m_2 \left(V_{f1} + V_{i1} - V_{i2} \right)$$

$$V_{i1}(m_1 - m_2) + V_{i2}(m_2 + m_2) = V_{f1}(m_1 + m_2)$$

$$\Rightarrow \quad \boxed{V_{f1} = \frac{m_1 - m_2}{m_1 + m_2} V_{i1} + \frac{2 m_2}{m_1 + m_2} V_{i2}}$$

Recall: $V_{f2} = V_{f1} + V_{i1} - V_{i2}$

$$\Rightarrow \quad V_{f2} = \left[\frac{m_1 - m_2}{m_1 + m_2} + 1 \right] V_{i1} + \left[\frac{2 m_2}{m_1 + m_2} - 1 \right] V_{i2}$$

$$= \frac{m_1 - \cancel{m_2} + m_1 + \cancel{m_2}}{m_1 + m_2} V_{i1} + \frac{2 m_2 - m_1 - m_2}{m_1 + m_2} V_{i2}$$

$$\boxed{V_{f2} = \frac{m_2 - m_1}{m_1 + m_2} V_{i2} + \frac{2 m_1}{m_1 + m_2} V_{i1}}$$

Note: symmetric w.r.to "1" $\leftrightarrow$ "2" interchange

E.g. when $V_{i2} = \emptyset$, $V_{i1} = V_0$, $m_1 = m_2$ (the billiard balls)

$$\left. \begin{aligned} V_{f1} &= \frac{m_1 - m_2}{m_1 + m_2} V_0 + \emptyset = \emptyset = V_{i2} \\ V_{f2} &= \frac{2 m_1}{m_1 + m_2} V_0 = V_0 = V_{i1} \end{aligned} \right\} \text{ as expected!}$$

Special cases:

① $m_1 = m_2 = m$ (billiard)

$\Rightarrow$ $V_{f1} = V_{i2}$, $V_{f2} = V_{i1}$ (i.e. masses exchange velocities)

② $m_2 \gg m_1$ (a "ball" hits a "wall")

$$\begin{cases} V_{f1} \simeq V_{i1} \dfrac{-m_2}{m_2} + V_{i2} \dfrac{2 m_2}{m_2} = -V_{i1} + 2 V_{i2} \\ V_{f2} \simeq V_{i2} \quad \text{(i.e. the "wall" is not affected)} \end{cases}$$

In particular, if $V_{i2} = V_{f2} = \emptyset$ $\Rightarrow$ $V_{f1} = -V_{i1}$ ("ball" bounces)

③ $m_1 \gg m_2$ (a golf club hits a ball)

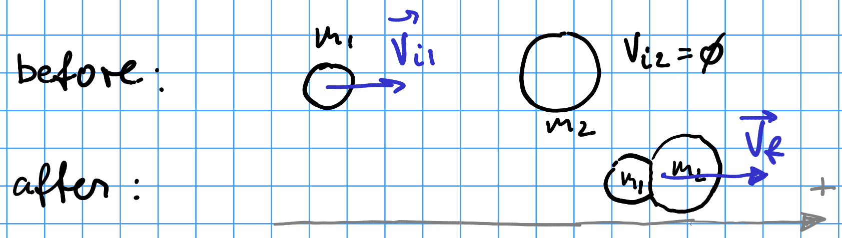
$V_{f1} \simeq V_{i1}$ (club swings through unaffected)

$$V_{f2} \simeq V_{i1}\frac{2m_1}{m_1} + V_{i2}\frac{-m_1}{m_1} = 2V_{i1} - V_{i2}$$

In particular, if $V_{i2} = \emptyset$ (most golf balls are stationary)

$\Rightarrow V_{f2} \simeq 2V_{i1}$ or $V_{ball} = \underline{\underline{2}}\, V_{club}$

Ex. an inelastic collision in 1D



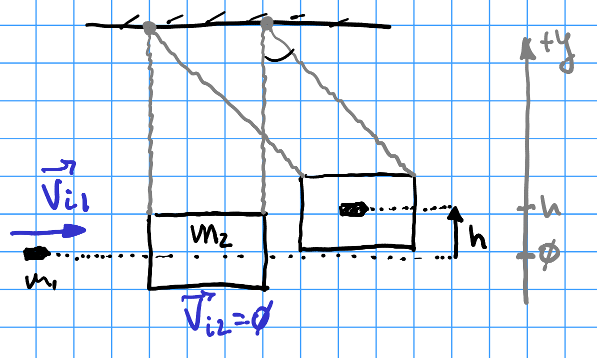
(stick & move together)

Note: $\Delta E \neq \emptyset \rightarrow$ ~~C.o.K.E.~~

But: C.o.M. applies: $m_1\vec{V}_{i1} + \cancel{m_2\vec{V}_{i2}}^{\,\emptyset} = (m_1+m_2)\vec{V}_f$

$$\vec{V}_f = \vec{V}_{i1}\underbrace{\frac{m_1}{m_1+m_2}}_{\text{a scalar!}} \Rightarrow \vec{V}_f \parallel \vec{V}_{i1}$$

Ex a ballistic pendulum



- only C.o.M in the collision:

$$V_{i1} = V_f\frac{m_1+m_2}{m_1}$$

- CoE after the collision:

$$\frac{1}{2}\cancel{(m_1+m_2)}V_f^2 = \cancel{(m_1+m_2)}gh$$

$$V_f = \sqrt{2gh}$$

$$\Rightarrow V_{i1} = \sqrt{2gh}\,\frac{m_1+m_2}{m_1}$$

E.g. in the lab: $V_{i1} = \sqrt{2 \times 9.80 \times 0.650}\,\frac{0.0100 + 2.50}{0.0100} \simeq 896\,\frac{m}{s}$

Ex a superball bounce. How high?

- a challenge problem, submit solutions via PDF for a bonus.

Angular momentum, L

By analogy with translational motion

$L = I\omega$ $\qquad\qquad$ $p = mv$

$\tau = \dfrac{\Delta L}{\Delta t}$ $\qquad\qquad$ $F = \dfrac{\Delta p}{\Delta t}$

if $\tau = \phi \Rightarrow L = \text{const}$ $\qquad$ if $F = \phi \Rightarrow p = \text{const}$

Ex. a spinning skater

no friction $\Rightarrow$ all forces / torques are internal

$\Rightarrow L = \text{const} = I\omega \quad \Rightarrow \quad I_i \omega_i = I_f \omega_f$

But: $I \propto mr^2$, as the arms are brought in,

$\underline{I \text{ decreases}} \Rightarrow \omega\ \underline{\text{increases}} \Rightarrow \omega_f = \omega_i \dfrac{I_i}{I_f} > \omega_i$

Ex. skater reduces her I by $\frac{1}{2}$ $\Rightarrow$ ω increases by $\times 2$

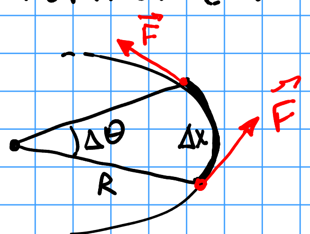
But: K_r (rotational KE) is <u>not</u> conserved!

$K_r = \frac{1}{2} I \omega^2 \longrightarrow K_r' = \frac{1}{2} I' (\omega')^2$

$= \frac{1}{2}\left(\frac{1}{2} I\right)(2\omega)^2 = \frac{1}{2} I \omega^2 \left(\times \frac{1}{2} \times 2^2\right)$ $\quad 2$

$K_r' = 2 K_r$ $\qquad$ KE doubled !? $\qquad$ (work done by moving arms/legs in)

- rotational work and power



$W = F\Delta x = F(R\Delta\theta) = \underbrace{FR}_{\tau}\, \Delta\theta = \tau \Delta\theta$

Again: $\quad W = \tau\Delta\theta \longleftrightarrow W = F\Delta x$

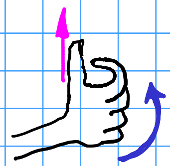
Power = rate of doing work, $P = \dfrac{W}{\Delta t}$

$\Rightarrow \quad P_{tr.} = \dfrac{F\Delta x}{\Delta t} = Fv \quad \longleftrightarrow \quad P_{rot.} = \dfrac{\tau\Delta\theta}{\Delta t} = \tau\omega$

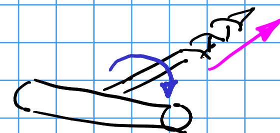
- vector nature of $\vec{L}$ and $\vec{\tau}$

$$L = I\omega \qquad \text{vs} \qquad \vec{p} = m\vec{v}$$

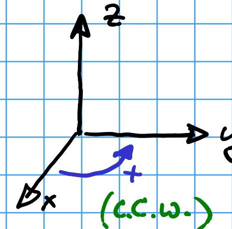
the direction of $\vec{L}$ and $\vec{\tau}$ = ? a convention



r. hand

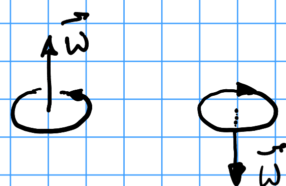


r.h. corkscrew

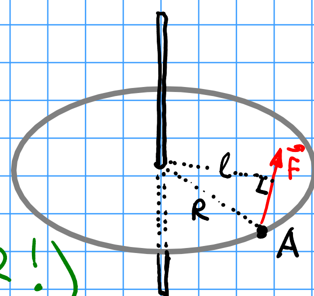


looking down from +z
x → y is +ve

⇒ the right-hand rule



$\tau = F\ell$
(not FR!)



R = distance to A
ℓ = distance to the line of $\vec{F}$ applied at A

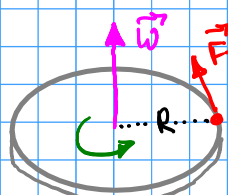
$\vec{L} = I\vec{\omega}$ ⇒ $\vec{L} \uparrow\uparrow \vec{\omega}$
involves only m, r^2 → a scalar

Recall : $\vec{F}$ changes $\vec{p}$

$$\left.\begin{aligned}\Delta\vec{p} &= \Delta(m\vec{v}) = m\,\Delta\vec{v}\\ \vec{F} &= m\frac{\Delta\vec{v}}{\Delta t} = \frac{m\,\Delta\vec{v}}{\Delta t} = \frac{\Delta\vec{p}}{\Delta t}\end{aligned}\right\} \quad \vec{F} = \frac{\Delta\vec{p}}{\Delta t} \quad \text{or} \quad \boxed{\Delta\vec{p} = \vec{F}\Delta t}$$

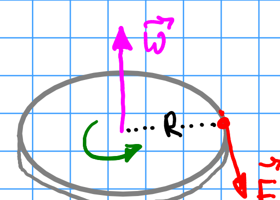
Similarly, $\vec{\tau}$ changes $\vec{L}$:

$$\left.\begin{aligned}\Delta\vec{L} &= \Delta(I\vec{\omega}) = I\,\Delta\vec{\omega}\\ \vec{\tau} &= I\vec{\alpha} = I\frac{\Delta\vec{\omega}}{\Delta t} = \frac{I\,\Delta\vec{\omega}}{\Delta t} = \frac{\Delta\vec{L}}{\Delta t}\end{aligned}\right\} \quad \vec{\tau} = \frac{\Delta\vec{L}}{\Delta t} \quad \text{or} \quad \boxed{\Delta\vec{L} = \vec{\tau}\Delta t}$$



⇒ $\vec{\tau}$

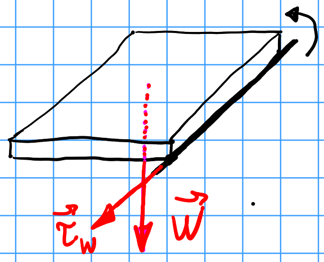
$\vec{\tau}$ along $\vec{L}$
$|\vec{L}|$ increases



⇒ $\vec{\tau}$

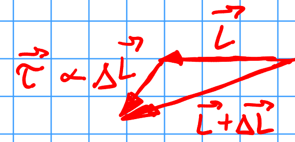
$\vec{\tau}$ against $\vec{L}$
$|\vec{L}|$ decreases

- In general, $\vec{\tau}$ and $\vec{L}$ need not be colinear, e.g. $\vec{\tau} \perp \vec{L}$ will not change $|\vec{L}|$ but change the direction.

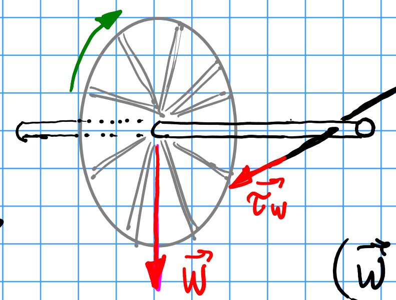


a hinged board
$(\vec{W} = \text{weight})$

vs



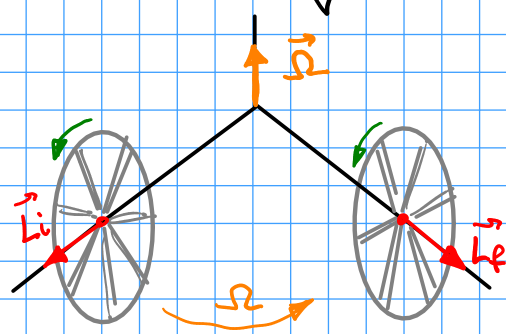
Note: $\Delta\vec{L} \propto \vec{\tau} \perp \vec{L}$
($\vec{\tau}$ "rotates" $\vec{L}$)



a hinged gyro

$(\vec{W} = \text{weight})$

$\Downarrow$



precession

$\vec{\Omega}$ = precession of the axis of spinning