Archimedes' principle

Everybody knows the story's punch line: A man is so excited by the idea that came to him in a bathtub that he runs naked to the emperor's palace, shouting "Eureka!" But what exactly was the idea that made Archimedes forget the dress code?

Survey your friends who are not taking this physics course, and most are likely to respond with some description of *buoyancy*: A body immersed in a fluid experiences a buoyant force equal to the weight of the displaced fluid.

That's important enough, and Archimedes did state a very concise formulation of the buoyancy principle. However, boats floated long before Archimedes came along. Why would a better formulation of an old idea excite him so?

The full story of "Eureka!" is somewhat more subtle. Archimedes figured out a way to *use* the buoyant force to solve a very important practical problem of catching the crooks who were defrauding the treasury by passing off gold-silver alloy coins as pure gold ones. An alloy is a material composed of two or more different metals.

Before you continue reading the next paragraph, spend a few minutes trying to think of a solution to this challenge. It's not an easy one!

Archimedes' solution (which brought him both the satisfaction of resolving a intellectual challenge and a considerable monetary reward) could be implemented quickly and easily and required only the simplest of tools: A balance scale, weights made of pure silver and pure gold, and a tub of water.

First, you had to use the weights made of gold balance out the unknown material. Then you would submerge both sides of the balance in water. If the two arms remained balanced, then the unknown material was also gold since the same mass of the material displaced the same volume of water on both sides and thus both sides experienced the same buoyant force equal to the weight of that water.

If, however, the material was not really gold, its density was slightly different from that of the pure gold, and the same mass would displace a different volume of water. The buoyant force would be slightly different on the two sides of the balance scale, and the submerged balance would tilt.

In fact, by replacing the pure gold weights with a mix of gold and silver weights and adjusting their ratio until the balance scale remained level in and out of water, one could measure the exact make-up of the alloy — and catch the crooks!

Density of materials

The *density* ρ of a material is defined as the ratio of its mass m to its volume V,

$$\rho = \frac{m}{V} \tag{1.1}$$

and has units such as $\mathrm{kg/m^3}$ or $\mathrm{g/cm^3}$. The density of an alloy can be determined by dividing the sum of the component masses by the sum of the component volumes. For an alloy of aluminum (Al) and copper (Cu), the alloy density is given by:

$$\rho_{\mathrm{alloy}} = \frac{m_{\mathrm{Al}} + m_{\mathrm{Cu}}}{V_{\mathrm{Al}} + V_{\mathrm{Cu}}}. \tag{1.2}$$

Specific gravity of materials

The *specific gravity* $\mathcal{S}_x$ of a material x is defined as the dimensionless (unitless) ratio of the density of the material ρ_x to the density of water $\rho_{\mathrm{H_2O}}$:

$$\mathcal{S}_x = \frac{\rho_x}{\rho_{\mathrm{H_2O}}} \tag{1.3}$$

Suppose that you have a solid object of mass m_x and volume V_x that weighs $w_{\mathbf{a}} = m_x g$ at the earth's surface and a container full of water so that their combined weight is $w_{\mathbf{b}}$. If you submerge the object in the container, some of the water will spill out of the container, decreasing the combined weight of object and container to $w_{\mathbf{c}}$.

The volume of the spilled water $V_{\mathrm{H_2O}}$ is equal to the volume of the object: $V_{\mathrm{H_2O}} = V_x$. The weight of the spilled water, or the apparent weight loss of the object when submerged in water, is $m_{\mathrm{H_2O}}\, g = w_{\mathbf{b}} - w_{\mathbf{c}}$.

If the numerator and denominator of Equation 1.3 are multiplied by $V g$, the numerator becomes the weight of the object and the denominator becomes the weight of the water displaced. This provides us with a practical expression for the object's specific gravity:

$$\mathcal{S}_x = \frac{m_x g}{m_{\mathrm{H_2O}}\, g} = \frac{w_{\mathbf{a}}}{w_{\mathbf{b}} - w_{\mathbf{c}}} = \frac{\text{weight of object in air}}{\text{apparent weight loss when submerged in water}} \tag{1.4}$$

Determining the alloy mass ratio from specific gravity results

To compare a theoretical result, the mass ratio of the two metals, to that obtained from the experimentally determined specific gravities of the component materials, Equation 1.2 can be re-arranged as follows:

$$\begin{aligned}
m_{\mathrm{Al}} + m_{\mathrm{Cu}} &= \rho_{\mathrm{alloy}}\,(V_{\mathrm{Al}} + V_{\mathrm{Cu}}) \\
&= \rho_{\mathrm{alloy}} \left(\frac{m_{\mathrm{Al}}}{\rho_{\mathrm{Al}}} + \frac{m_{\mathrm{Cu}}}{\rho_{\mathrm{Cu}}}\right) \\
&= \mathcal{S}_{\mathrm{alloy}}\,\rho_{\mathrm{H_2O}} \left(\frac{m_{\mathrm{Al}}}{\mathcal{S}_{\mathrm{Al}}\,\rho_{\mathrm{H_2O}}} + \frac{m_{\mathrm{Cu}}}{\mathcal{S}_{\mathrm{Cu}}\,\rho_{\mathrm{H_2O}}}\right) \\
&= m_{\mathrm{Al}} \left(\frac{\mathcal{S}_{\mathrm{alloy}}}{\mathcal{S}_{\mathrm{Al}}}\right) + m_{\mathrm{Cu}} \left(\frac{\mathcal{S}_{\mathrm{alloy}}}{\mathcal{S}_{\mathrm{Cu}}}\right)
\end{aligned}$$

and is finally expressed in terms of the mass ratio:

$$\frac{m_{\mathrm{Al}}}{m_{\mathrm{Cu}}} = -\frac{1 - \mathcal{S}_{\mathrm{alloy}}/\mathcal{S}_{\mathrm{Cu}}}{1 - \mathcal{S}_{\mathrm{alloy}}/\mathcal{S}_{\mathrm{Al}}}. \tag{1.5}$$

The corresponding error equation is:

$$\delta\left(\frac{m_{\mathrm{Al}}}{m_{\mathrm{Cu}}}\right) = \left(\frac{m_{\mathrm{Al}}}{m_{\mathrm{Cu}}}\right) \sqrt{2\left(\frac{\delta\mathcal{S}_{\mathrm{alloy}}}{\mathcal{S}_{\mathrm{alloy}}}\right)^2 + \left(\frac{\delta\mathcal{S}_{\mathrm{Al}}}{\mathcal{S}_{\mathrm{Al}}}\right)^2 + \left(\frac{\delta\mathcal{S}_{\mathrm{Cu}}}{\mathcal{S}_{\mathrm{Cu}}}\right)^2} \tag{1.6}$$