Conservation of momentum

The velocity $\vec{v}$ of a body of mass m is a vector quantity, symbolized an arrowed line segment.

- The magnitude of the vector represents the speed of the body, a scalar quantity.
- The orientation of the vector in space represents the direction of motion of the body.

If we consider a collision of two objects with masses m_1 and m_2, and with velocities $\vec{v}_{1b}$ and $\vec{v}_{2b}$ *before* the collision, and velocities $\vec{v}_{1a}$ and $\vec{v}_{2a}$ *after* the collision, we note that it is generally difficult, if not impossible, to *predict* these resulting velocities $\vec{v}_{1a}$ and $\vec{v}_{2a}$.

To accomplish this, one would need to have a complete knowledge of the physical characteristics of the objects (size, shape, etc.) and of the geometry of the interaction. However, it is practical to measure all of the relevant quantities and then check to see whether their values are reasonable.

The linear momentum $\vec{p}$ of an object is a vector equal to the product of its mass m (a scalar) and its velocity $\vec{v}$ (a vector). For two colliding objects, the law of conservation of linear momentum states that:

> *If the net external force acting on the colliding objects is zero, then the total momentum $\vec{p}_b$ of the colliding objects before the collision is equal to the total momentum $\vec{p}_a$ after the collision.*

A mathematical formulation of this law for a collision between two objects with masses m_1 and m_2 is a vector equation and can be expressed as follows:

$$\begin{aligned} \vec{p}_{1b} + \vec{p}_{2b} &= \vec{p}_{1a} + \vec{p}_{2a} \\ m_1\vec{v}_{1b} + m_2\vec{v}_{2b} &= m_1\vec{v}_{1a} + m_2\vec{v}_{2a}. \end{aligned} \tag{4.1}$$

Rearranging Equation 4.1 in terms of the changes in the momentum $\Delta\vec{p}_1$ and $\Delta\vec{p}_2$ of the two objects shows that the net change will be zero and that these vectors will be oriented anti-parallel to one another:

$$\vec{p}_{1a} - \vec{p}_{1b} + \vec{p}_{2a} - \vec{p}_{2b} = 0 \quad \rightarrow \quad (\vec{p}_{1a} - \vec{p}_{1b}) = -(\vec{p}_{2a} - \vec{p}_{2b}) \quad \rightarrow \quad \Delta\vec{p}_1 = -\Delta\vec{p}_2$$

Similarly, the changes in the velocity for the two objects will result in two anti-parallel vectors:

$$m_1(\vec{v}_{1a} - \vec{v}_{1b}) = -m_2(\vec{v}_{2a} - \vec{v}_{2b}) \quad \rightarrow \quad m_1\Delta\vec{v}_1 = -m_2\Delta\vec{v}_2 \tag{4.2}$$

Since the two vectors $\Delta\vec{v}_1$, $\Delta\vec{v}_2$ are aligned, the vector Equation 4.2 can be reduced to a scalar equation and expressed in terms of the magnitudes of the vector changes in velocities $|\vec{v}_{2a} - \vec{v}_{2b}|$ and $|\vec{v}_{1a} - \vec{v}_{1b}|$. That is, the magnitudes of the vectors on each side of Equation 4.2 are equal:

$$m_1|\Delta\vec{v}_1| = m_2|\Delta\vec{v}_2| \tag{4.3}$$

Rearranging Equation 4.3 as a ratio of masses m_1 and m_2, we obtain:

$$\frac{m_1}{m_2} = \frac{|\Delta\vec{v}_2|}{|\Delta\vec{v}_1|} = \frac{|\vec{v}_{2a} - \vec{v}_{2b}|}{|\vec{v}_{1a} - \vec{v}_{1b}|}. \tag{4.4}$$

Thus, measuring the masses of the two pucks and the velocities of the pucks before and after the collision to see if Equation 4.4 is satisified is an experimental test of the law of conservation of linear momentum.

Conservation of energy

It is also of interest to know whether kinetic energy $K = mv^2/2$ is conserved during a collision. The "quality factor" $Q = K_a/K_b$ is defined as the ratio of the total kinetic energy K_a after the collision to the total kinetic energy K_b before.

The kinetic energy, and therefore Q, are scalar quantities.

- In an *elastic* collision, the kinetic energy of the system is conserved, and so $Q = 1$ and

$$m_1\, v_{1b}^2 + m_2\, v_{2b}^2 = m_1\, v_{1a}^2 + m_2\, v_{2a}^2 \tag{4.5}$$

- in an *inelastic* collision some of the kinetic energy may be transformed into heat and sound energy during the collision, or there could be frictional forces acting on the masses during the interaction, and so $Q < 1$.

- In principle, in a *superelastic* collision, the total translational kinetic energy may even increase ($Q > 1$), for example if some *rotational* kinetic energy imparted onto the pucks before the collision transfers into translational kinetic energy, or if additional energy is released by the collision itself (*e.g.*, a collision of two spring-loaded mousetraps).

Note that Q depends on the square of the velocity and hence will be very sensitive to variations in v.

$$Q = \frac{K_a}{K_b} = \frac{\frac{1}{2}\, m_1\, v_{1a}^2 + \frac{1}{2}\, m_2\, v_{2a}^2}{\frac{1}{2}\, m_1\, v_{1b}^2 + \frac{1}{2}\, m_2\, v_{2b}^2} = \frac{m_1\, v_{1a}^2 + m_2\, v_{2a}^2}{m_1\, v_{1b}^2 + m_2\, v_{2b}^2} \tag{4.6}$$

Some interesting collisions trivia

In a one-dimensional elastic collision between two objects of masses m_1 and m_2, where the first object has a speed $v_{1b} = |\vec{v}_{1b}|$ and the second object is stationary so that $v_{2b} = 0$:

- if $m_1 = m_2$, then after the collision $v_{1a} = 0$ and $v_{2a} = v_{1b}$;
- if $m_1 \gg m_2$, then after the collision $v_{1a} \approx v_{1b}$ and $v_{2a} \approx 2v_{1b}$;

These results can be obtained from the conservation of energy and momentum Equations 4.5 and 4.1, setting $v_{2b} = 0$ then solving each for v_{1a} and equating the two equations to get

$$v_{2a} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1b} \qquad \text{and from this} \qquad v_{1a} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1b}. \tag{4.7}$$

When $m_1 = m_2$, then $v_{2a} = v_{1b}$ and when $m_1 \gg m_2$, then $v_{2a} \approx 2v_{1b}$, as expected.

Consider a two-dimensional elastic collision between two identical round objects of equal mass $m_1 = m_2$ that are not spinning. The first object has a velocity v_{1b} and the second object is stationary so that $v_{2b} = 0$. Then

- the angle between the velocity vectors $\vec{v}_{1a}$ and $\vec{v}_{2a}$ after the collisions is 90°.

This result can be proved as follows. Because the masses of the two objects are equal, we can set $m_1 = m_2$ in Equation 4.1 (momentum conservation) to obtain

$$\vec{v}_{1b} = \vec{v}_{1a} + \vec{v}_{2a}$$

This means that the three vectors in the previous equation form a triangle. One can see that the triangle is a right triangle by setting $m_1 = m_2$ in Equation 4.5 (conservation of kinetic energy for an elastic collision) to obtain

$$v_{1b}^2 = v_{1a}^2 + v_{2a}^2.$$

Because the side lengths of the triangle are related by the theorem of Pythagoras, it follows that the triangle is a right-angled triangle. Thus, the angle between the outgoing velocity vectors, $\vec{v}_{1a}$ and $\vec{v}_{2a}$, is $90°$. This completes the argument.

Billiards enthusiasts will know that this is true based on their experience. Certainly the situation with billiard balls is more complicated, because a skilled practitioner can cause the cue ball to spin in various ways, but if the spin of the cue ball is minimal then the result is approximately true on the billiard table.