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Physics 2P20
Introductory Mechanics

Physics Department

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1.1 Introduction

Some degree of uncertainty exists in any measurement. This uncertainty is called the ERROR. It is to be distinguished from the notion of a mistake and it is not the discrepancy between the student-measured value and that given in a textbook. Rather, it is a quantitative indication of the probability that a further measurement under the same conditions would give a result that falls within a specified range of the reported value. The determination of the error associated with a measurement is part of the general problem of data analysis. In this introductory session we can only hope to touch upon a few of the important concepts.

1.2 Instrumental Error and Significant Figures

An error may be associated with a single measurement, in which case it is a reflection of the limitations on the precision of the measuring device.

Consider a measurement of time, using a stopwatch on which the smallest scale division is 0.2 s. If a single reading of, say, the time taken for a glider to travel over a marked length of an air track is determined as 9.2 s, then it is properly recorded as

$$t = 9.2 \pm 0.1 \text{ s.} \tag{1.1}$$

Equation (1.1) indicated that the observed time lies closer to 9.2 than to 9.0 or 9.4 s, but could have any value between 9.1 and 9.3 s; see Figure 1.1(a).

In this case, the uncertainty in the time is due to the limitation of the instrument; the *instrumental error* δt is then

$$\delta t = \pm 0.1 \text{ s},$$

and the reported value has two significant figures.

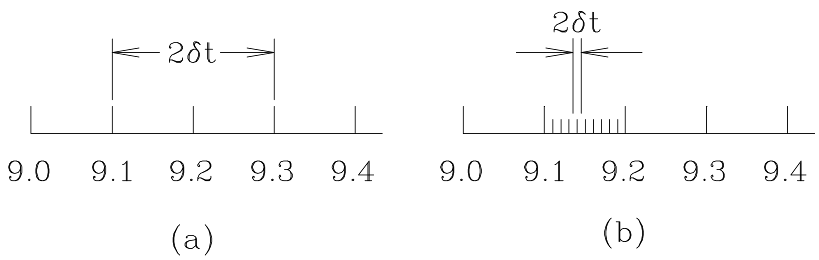


Figure 1.1: Comparison of instrumental error ((a) stopwatch vs. (b) electronic timer)

Assume the stopwatch is replaced with an electronic timer, automatically started and stopped as the glider passes two selected points on the air track. If the smallest time division is now 0.01 s, the result of a measurement might be recorded as

$$t = 9.140 \pm 0.005 \text{ s.} \tag{1.2}$$

The number of significant figures is now 3, reflecting the smaller instrumental error of $\delta t = 0.005$ s. Thus, as shown in Figure 1.1(b), Equation (1.2) indicates that the observed time lies between 9.135 and 9.145 s.

With some instruments, a result is read from a continuously marked dial (this is an "analogue" read-out while the former is a "digital" read-out). In such cases an investigator may be able to decrease the instrumental uncertainty below the limit imposed by the smallest scale division by interpolation.

Table #1		**Table #2**	
Trial #1	**Length (cm)**	**Trial #2**	**Time (seconds)**
1	88.90	1	1.88
2	88.88	2	1.86
3	88.92	3	1.88
4	88.91	4	1.90
5	88.89	5	1.88
6	88.90	6	1.92
7	88.88	7	1.88
8	88.91	8	1.90
9	88.91	9	1.90
10	88.93	10	1.90
11	88.89		
12	88.91		

(!) **Exercise #1.**
In an experiment the value of the gravitational acceleration g is measured by determining the period T of a pendulum of length L. g can be calculated from L and T according to:

$$g = \frac{4\pi^2 L}{T^2}.$$

Many independent measurements of L and T (for the same pendulum) are made; the data collected are shown in Tables 1 and 2.

[?] What instrumental errors would you infer for the measurements of the length and the time?

1.3 Random Errors

In those cases in which only a single measurement is possible, the error reported is necessarily the instrumental error described in Section 1.2.

If the situation permits, however, it is good practice to repeat a measurement several times. If the precision of the instrument is low, it may be found that all measurements are found to be the same within the large instrumental error.

On the other hand, if the instrumental error is reduced by going to a more precise measuring device, it is frequently found that the values so determined differ from one another by more than the instrumental error and these differences generally become more noticeable as the accuracy of the instrument increases. The differences may arise from a number of causes, all of which are characterized by the fact that within the given conditions of the measurement there are random or experimentally uncontrollable variations. One of these, for example, may be due to variation in the judgement of the investigator if he uses interpolation between smallest scale divisions. In other cases, it may arise from slight temperature changes or vibrations, or lack of humidity control, *etc.* Finally, in an important class of measurements, the parameter concerned may have an intrinsic variability; for example, the number of radioactive atoms, in a given sample N_0, that decay during a time interval t, or the number of electrons emitted per unit time from a filament at temperature T, and so on.

In all of these cases the resultant uncertainty in the reported value of the quantity measured can best be represented in statistical terms.

Assume we wish to determine the value of some quantity q and we set out to do so by making a set of n measurements, all made under identical conditions so far as they may be controlled, and all having the same associated instrumental error. We call such a set a "sample." It may be shown (and is intuitively "obvious") that the single parameter that best represents the sample is the arithmetic mean, defined by

$$\langle q \rangle = \frac{1}{n} \sum_{j=1}^{n} q_j. \tag{1.3}$$

The individual q_j's, of course, contain the number of significant figures consistent with the instrumental error.

Stating $\langle q \rangle$ alone, however, is not sufficient to characterize the sample of n measurements. This can be appreciated intuitively by reference to Figure 1.2, in which the results of three sets, each of n measurements, all having the same average value, are represented as frequency histograms. These are constructed by plotting along the ordinate axis the number of times n_i the measured quantity q falls in the range $q_1 \pm \Delta q_i$, against the associated value of q_i plotted along the abscissa axis. The latter is marked off in segments or "bins" of width $2\delta q$. It is clear that if we are to represent the n measurements by the average $\langle q \rangle$, we require to indicate, quantitatively, the degree of "spread" of values about $\langle q \rangle$.

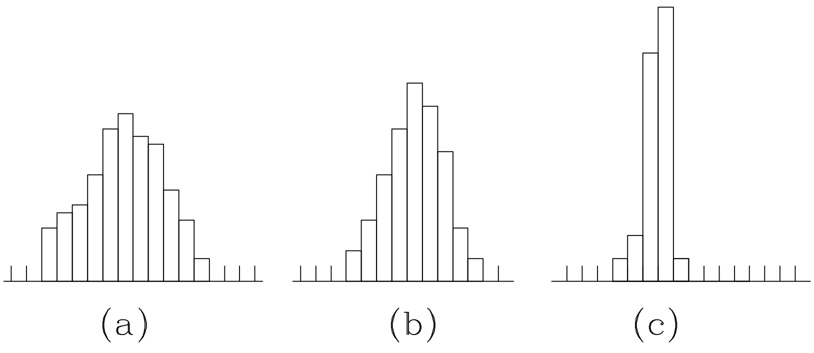


Figure 1.2: Histograms with Gaussian shape

Exercise #2.
From the data in Tables 1 and 2, calculate $\langle L \rangle$ and $\langle T \rangle$. Construct histograms for L and T.

The shape of the histograms shown in Figure 1.2 is called a Gaussian; it arises whenever the variations about the mean are random. We shall restrict ourselves to such cases, which are common, but by no means unique.

Since the individual values of q are distributed about $\langle q\rangle$, we can define the *deviation* of say the j^{th} value q_j from the mean as

$$\Delta q_j = (q_j - \langle q\rangle). \tag{1.4}$$

We might be tempted to use the average of the deviations, *i.e.*,

$$\langle\Delta\rangle = \frac{1}{n}\sum_{j=1}^{n}\Delta q_j,$$

as a measure of the spread of q values in our sample. However, as it will be shown in Exercise 3, this is not a fruitful approach. Instead, we introduce the *DISPERSION* of the sample as a measure of the spread, defined as

$$\langle(\Delta q)^2\rangle = \frac{1}{n}\sum_{j-1}^{n}(q_j - \langle q\rangle)^2. \tag{1.5}$$

A linear measure of the spread for the sample of n measurements is given by the *STANDARD DEVIATION OF THE SAMPLE*, S, defined by

$$S = \sqrt{\langle(\Delta q)^2\rangle}. \tag{1.6}$$

(!) **Exercise #3.**
Show that the average deviation $\langle\Delta q_j\rangle$ is always zero. Calculate the standard deviation of the sample, S from the data in Tables 1 and 2; call the results S_L and S_T.

S is a measure of the width of the histogram; a large S value corresponds to a wide spread of values around the average value. If one further measurement of q was done, there is a 68% probability that it will fall in the range $\langle q\rangle \pm S$, a 95% chance that it will fall in the range $\langle q\rangle \pm 2S$. S is therefore an indication of the precision of a *single* measurement.

In principle, however, we want a measure of the probability that our sample mean q lies within a certain range of the "true value" which we'd obtain if we were to make many, many repeated *samples*, calculate the mean in each case and see how they are distributed. The standard deviation of that distribution is called the *STANDARD DEVIATION OF THE MEAN*, σ_m, and statistical theory shows we can calculate σ_m from S and n by:

$$\sigma_m = \frac{S}{\sqrt{n-1}}. \tag{1.7}$$

Clearly, as we increase the number of measurements in our sample, the mean of that sample will be closer to the true value.

Note this difference: S gives the precision of *a single* measurement, whereas σ_m gives the precision of the *average* of *many* (n) measurements. Obviously, the average value is more precise than the value obtained from one measurement; therefore σ_m is smaller than S.

(!) **Exercise #4.**
Calculate $\sigma_m(L)$ and $\sigma_m(T)$.

In summary, the error of a single measurement is determined by the precision of the instrument. The error of the result from a sample of n measurements, expressed as the mean $\langle q\rangle$, is expressed in terms of σ_m and quoted as

$$q = \langle q\rangle \pm \sigma_m.$$

If the distribution of q values about the mean is Gaussian, then the probability that $\langle q\rangle$ lies within $\pm\sigma_m$ of the "true" value is 68.3%; the probability that it lies within $\pm 2\sigma_m$ of the "true" value is 95.45%, *etc.*

1.4 The Propagation of Errors

Very frequently a parameter of interest is not measured directly, but is deduced from one or more parameters that are measured, through some functional relationship. For example, the gravitational constant g is approximately given in terms of length L of a pendulum and the period T of a swing by

$$g = \frac{4\pi^2 L}{T^2}.$$

If each of the latter have errors associated with them, we require to know how these errors are propagated to give rise to a corresponding uncertainty in the parameter of interest.

In the following we shall consider a general case, in which the quantity Z is related to two directly measureable parameters, x, y, in the form

$$Z = Z(x, y). \tag{1.8}$$

We assume $\langle x \rangle$ and σ_x have been determined, as in Section 1.3, through n_x measurements, and $\langle y \rangle$ and σ_y through n_y measurements. The reported value of Z is then that obtained from

$$\langle Z \rangle = Z(\langle x \rangle, \langle y \rangle). \tag{1.9}$$

(!) **Exercise #5.**
Calculate the average value of g.

We need to find σ_Z, the uncertainty in Z arising from σ_x and σ_y. Consider $Z_{jk} = Z(x_j, y_k)$, *i.e.*, the Z value associated with the j^{th} measured value of x and the k^{th} measured value of y. The deviation of Z_{jk} from the mean value is then

$$\Delta Z_{jk} = (Z_{jk} - \langle Z \rangle). \tag{1.10}$$

Now by definition,

$$\Delta x_j = x_j - \langle x \rangle, \quad \Delta y_k = y_k - \langle y \rangle.$$

Thus

$$x_j = \langle x \rangle + \Delta x_j, \quad y_k = \langle y \rangle + \Delta y_k,$$

and

$$Z_{jk} = Z(\langle x \rangle + \Delta x_j, \quad \langle y \rangle + \Delta y_k). \tag{1.11}$$

Z_{jk} can be expanded as a Taylor's series about $Z(\langle x \rangle, \langle y \rangle)$ in the form:

$$\begin{aligned} Z_{jk} &= Z(\langle x \rangle + \Delta x_j, \langle y \rangle + \Delta y_k) \\ &= Z(\langle x \rangle, \langle y \rangle) + \left(\frac{\partial Z}{\partial x}\right) \Delta x_j + \left(\frac{\partial Z}{\partial y}\right) \Delta y_k + \ldots . \end{aligned} \tag{1.12}$$

Here $(\partial Z/\partial x)$ is the partial derivative of Z with respect to x, y being held constant during differentiation. The resulting function is then evaluated at $(\langle x \rangle, \langle y \rangle)$. Similar remarks apply to $(\partial Z/\partial y)$.

(!) **Exercise #6.**
Evaluate $(\partial g/\partial L)_T$ and $(\partial g/\partial T)_L$ from $g = 4\pi^2 L/T^2$. Calculate the numerical values of these partial derivatives for $L = \langle L \rangle$ and $T = \langle T \rangle$.

Substituting (1.12) into (1.10) gives

$$\Delta Z_{jk} = \left(\frac{\partial Z}{\partial x}\right) \Delta x_j + \left(\frac{\partial Z}{\partial y}\right) \Delta y_k.$$

To obtain the dispersion in Z, i.e. $\langle(\Delta Z)^2\rangle$, we require

$$(\Delta Z_{jk})^2 = \left(\frac{\partial Z}{\partial x}\right)^2 (\Delta x_j)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 (\Delta y_k)^2 + 2\left(\frac{\partial Z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\Delta x_j \Delta y_k. \tag{1.13}$$

We now sum over the x's and the y's to obtain the dispersion:

$$\langle(\Delta Z)^2\rangle = \frac{1}{n_x n_y}\sum_{j=1}^{n_x}\sum_{k=1}^{n_y}(\Delta Z_{jk})^2.$$

This double sum is a bit tricky; the result is:

$$\langle(\Delta Z)^2\rangle = \left(\frac{\partial Z}{\partial x}\right)^2 S_x^2 + \left(\frac{\partial Z}{\partial y}\right)^2 S_y^2. \tag{1.14}$$

The quantities S_x and S_y have been defined previously; recall from equations (1.5) and (1.6) that

$$S_x^2 = \langle(\Delta x)^2\rangle = \frac{1}{n_x}\sum_{j=1}^{n_x}(\Delta x_j)^2,$$

ditto for S_y. Now we define again the standard deviation of the sample as:

$$S_z = \sqrt{\langle \Delta Z^2\rangle} = \sqrt{\left(\frac{\partial Z}{\partial x}\right)^2 S_x^2 + \left(\frac{\partial Z}{\partial y}\right)^2 S_y^2}. \tag{1.15}$$

(!) **Exercise #7.**
From the previously calculated values of S_L, S_T, and the partial derivatives, calculate S_g.

Finally, the precision of the average value of Z, $\langle Z\rangle$, is given by

$$\sigma_z = \frac{S_z}{\sqrt{n_x n_y - 1}}. \tag{1.16}$$

(!) **Exercise #8.**
Calculate σ_g, and compare it to $\langle g\rangle$.

We can now extend this treatment to the case of Z being a function of many variables, $Z = Z(x, y, p, q, \ldots)$, and calculating S_z from $S_x, S_y, S_p, S_q \ldots$. The resulting formulae are rather cumbersome; useful approximations are as follows:

Case A. $Z = ax + by$, a and b are constants.
Equation (1.15) would give:

$$S_z = \sqrt{a^2\, S_x^2 + b^2\, S_y^2}.$$

The right-hand-side is less than

$$[a^2\, S_x^2 + 2ab\, S_x\, S_y + b^2\, S_y^2]^{\frac{1}{2}} = (a\, S_x + b\, S_y).$$

Therefore, a pessimistic estimate of S_z is

$$S_z \simeq aS_x + bS_y.$$

Generalization: if Z is a "sum-type" function of the variables x, y, p, q, like

$$Z = ax + by - kp + \cdots,$$

then

$$S_z \simeq |aS_2| + |bS_y| + |kS_p| + \cdots .$$

Case B. $Z = x^p\, y^q$, p and q are constants.
Then $\left(\frac{\partial Z}{\partial x}\right) = p\, x^{p-1}\, y^q$, $\left(\frac{\partial Z}{\partial y}\right) = x^p\, q\, y^{q-1}$, and equation (1.15) would give:

$$S_z = \sqrt{(p\, x^{p-1}\, y^q)^2\, S_x^2 + (p^x\, q\, y^{q-1})^2\, S_y^2}.$$

The right-hand-side is less than

$$\sqrt{(p\, x^{p-1}\, y^q)^2\, S_x^2 + 2(p\, x^{p-1}\, y^q)(x^p\, q\, y^{q-1})\, S_x\, S_y + (x^p\, q\, y^{q-1})^2\, S_y^2} = (p\, x^{p-1}\, y^q)\, S_x + (x^p\, q\, y^{q-1})\, S_y.$$

Therefore, a pessimistic estimate of S_z is:

$$S_z \simeq (p\, x^{p-1}\, y^q)\, S_x + (x^p\, q\, y^{q-1})\, S_y.$$

This can be written as :

$$\frac{S_z}{Z} = p\frac{S_x}{x} + q\frac{S_y}{y}.$$

Generalization: if Z is a "product-type" function of the variables x, y, r, s, like

$$Z = \frac{x^p\, y^q \ldots}{r^t\, s^m \ldots},$$

then

$$\frac{S_z}{Z} \simeq \left|p\frac{S_x}{x}\right| + \left|q\frac{S_y}{y}\right| + \left|t\frac{S_r}{r}\right| + \cdots .$$

In words: "For a 'sum-type' function the errors are additive; for a 'product-type' function the relative errors are additive."

(!) **Exercise #9.**
Calculate S_g from S_L and S_T using these approximations; compare the result with the exact result obtained in Exercise #7.

Experiment 2

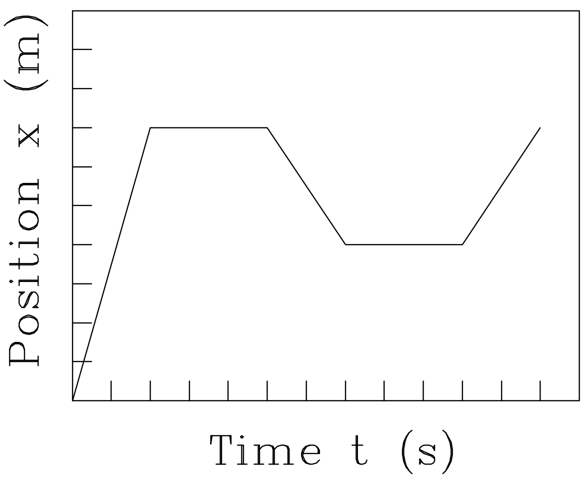
Data Analysis II

From the experiments you performed in Year-1 laboratories, you are already familiar with the data-acquisition interface and analysis tool called PhysicaLab. To review you may want to consult the lab manuals for the Year-1 courses (PHYS 1P91, 1P92, or 1P93), also available online from `www.physics.brocku.ca`. To reach Physica Online, point a browser to `www.physics.brocku.ca/physica`; even if you do not have a LabPro data acquisition interface available, you may simulate what happens in the lab by selecting a Demo mode from the Get Data screen.

PhysicaLab uses an ultrasonic range finder attached to a LabPro interface to track the position of a runner moving along an air track, as a function of time. The distance vs. time data can be saved into a file for later analysis, and later retrieved to be used in full interactive PHYSICA, or some other data analysis software (maple, gnuplot, *etc.*).

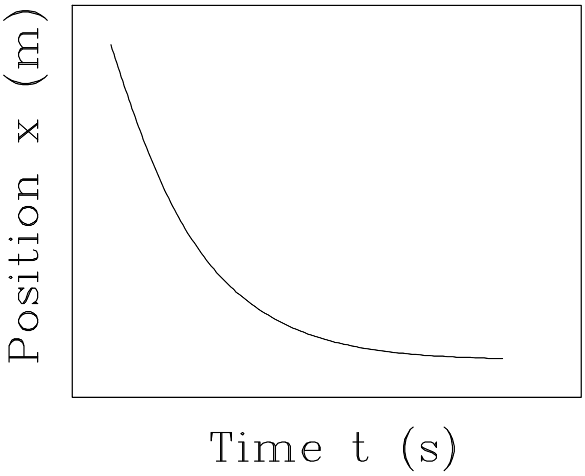
2.1 Analyse LabPro Data with PHYSICA

1. Using PhysicaLab and your hand, see if you can generate a similar graph to the one given below:



2. By applying a slight push to the air car, generate a trace of constant velocity. Make sure the air track is level first! Each student should generate their own trace, save it as a file. Then using PHYSICA they should determine the actual velocity (slope of x vs. t) from their graph.

3. Using a few sheets of folded paper on the track as a brake, generate a trace of deceleration. Plot the graph using PHYSICA and determine the average acceleration a. Determine the time at which braking first occurs, and at what time the car is stationary.



$$a_{avg} = \frac{\Delta v}{t} \tag{2.1}$$

The initial velocity is determined from the linear portion of the trace, while the deceleration time is measured from the first signs of slowing until the car is motionless.

2.2 Perform Least-Squares Analysis with PHYSICA

A student determines the "Moment of Inertia I^*" of a cradle-cylinder combination. The results are

d (in 10^{-2}m)	2.00	4.00	6.00	8.00	10.00
I^* (in 10^{-3}kg m^2)	3.26	4.55	6.307	8.768	12.31

Theoretically I^* is given by:

$$I^* = I_c + M_\ell R_\ell^2 + 2M_\ell d^2 \tag{2.2}$$

where I_c is the moment of inertia of the cradle, M_ℓ is the mass of the cylinder, and R_ℓ is the radius of the cylinder.

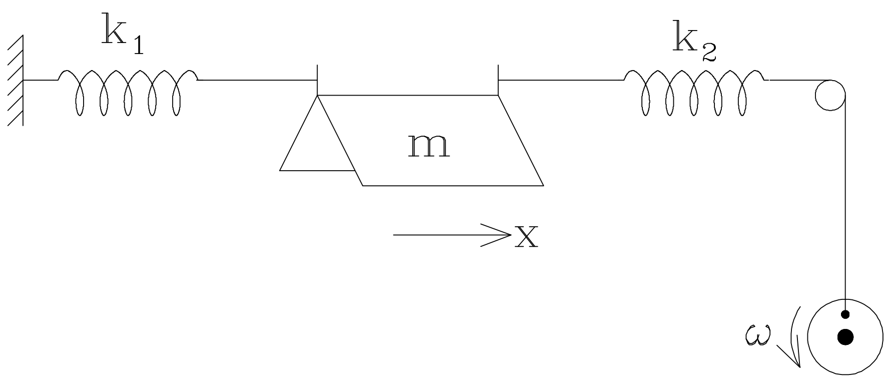
Equation (2.2) is of the form $y = ax + b$, with I^* corresponding to y, and d^2 corresponding to x.

Use PHYSICA to generate a properly labelled graph and also do a least-squares analysis, using the 5 data pairs, and from the result, calculate M_ℓ and R_ℓ ($I_c = 2.562 \times 10^{-3}$ kg m^2).

Experiment 3

Oscillators

The oscillating system consists of a car (mass m) on the Linear Air Track, connected to two stretched springs (spring constants k_1 and k_2); viscous damping can be provided by taping magnets to the sides of the car, and a periodic external force of variable frequency ω can be applied to m via a turntable. The position $x(t)$ of m can be either measured directly on the track, or a record of $x(t)$ can be graphed directly using PHYSICA. The period T (and therefore the frequency $\omega = 2\pi/T$) can be determined with a timer and a "flag" on the car.



(!) NOTE
Make sure to clean and level the air track prior to beginning the experiment for this and every subsequent lab that requires the use of the linear air track.

Please note that the sonar range finder has a minimum distance of 35 cm.

3.1 Undamped, Harmonic Oscillator

When the force F on the car is proportional to the displacement q from its equilibrium position, then the car will perform harmonic oscillations around $q = 0$. We then have for the force F,

$$F = -kq,$$

for the potential energy U,

$$U = \frac{1}{2}kq^2,$$

and for the differential equation of motion,

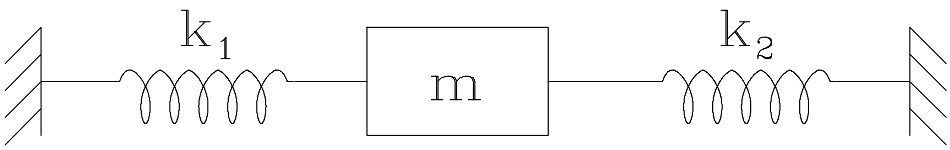
$$\frac{d^2q}{dt^2} + \omega_0^2 q = 0, \tag{3.1}$$

with $\omega_0^2 = k/m$. The motion is described by the solution to Equation (3.1),

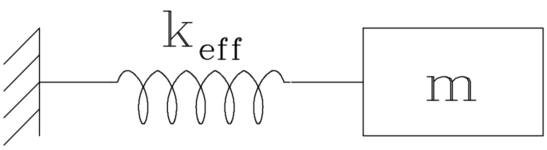
$$q = A\cos(\omega_0 t + \psi). \tag{3.2}$$

Amplitude A and phase angle ψ depend only on initial conditions, and are independent of ω_0.

(!) **Exercise #1.**
Remove the damping magnets from the sides and put them on top of the car. Carefully determine $\omega_0 \pm \sigma(\omega_0)$ (using the timer) of the oscillator as a function of amplitude, for the range $12 \geq A \geq 1$ cm. Determine m. Show theoretically, that the system



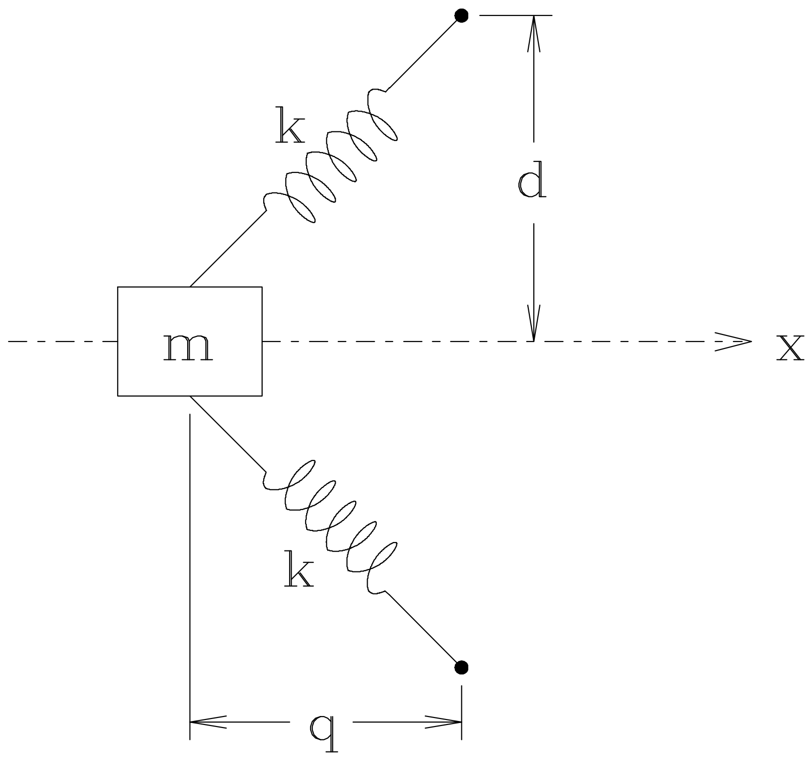
is equivalent to the system:



and derive an expression for k_{eff} in terms of k_1, k_2 and any other experimentally relevant parameters. Use the balance to determine m. From ω_0 and m, calculate k_{eff}.

3.2 Undamped, Anharmonic Oscillator

When the restoring force on the car is not exactly proportional to q, the car still oscillates around $q = 0$, but the oscillations will not be strictly harmonic, and amplitude and frequency are not independent. An example of an anharmonic oscillator is:



(!) **Exercise #2.**
Show that for the system above the restoring force (in the x-direction) on the car is

$$F = -kq \left[2 - 2 \left(\frac{\ell_0}{d\sqrt{1 + \frac{q^2}{d^2}}} \right) \right], \tag{3.3}$$

where ℓ_0 is the length of the relaxed spring, and $d > \ell_0$. Show that for small values of q/d this is approximately equal to:

$$F = -kq \left(2 - \frac{2\ell_0}{d} \right) - kq^3 \left(\frac{\ell_0}{d^3} \right). \tag{3.4}$$

Set up the oscillator as sketched above; make $d \cong 20$ cm. Carefully measure ω as a function of A adjusting d from 2 to 20 cm in 2 cm steps, and plot the results.

3.3 Damped, Harmonic Oscillator

Return to the set-up used in Section 3.1, but tape the damping magnet to the side of the car. This will provide a retarding, "frictional" force proportional to the velocity v of the car. The differential equation of motion is now:

$$m\frac{d^2q}{dt^2} = -kq - bv, \tag{3.5}$$

or

$$\frac{d^2q}{dt^2} + \gamma\frac{dq}{dt} + \omega_0^2 q = 0, \tag{3.6}$$

with $\gamma = b/m$ and $\omega_0^2 = k/m$.

For reasonably weak damping ($\gamma < 2\omega_0$) the motion of the car is still an oscillation, but with a frequency $\omega_1 (\neq \omega_0)$ and a steadily decreasing amplitude:

$$q(t) = A \exp\left(\frac{-\gamma t}{2}\right) \cos(\omega_1 t + \psi), \tag{3.7}$$

with

$$\omega_1^2 = \omega_0^2 - \left(\frac{\gamma}{2}\right)^2. \tag{3.8}$$

(!) **Exercise #3.**
With the magnet taped to the top of the car, weigh the car and determine ω_0. Take this measurement three times and use standard deviation to determine $\sigma(\omega_0)$.

Place the magnet on the side of the car, set the car in a damped oscillation, and measure $\omega_1 \pm \sigma(\omega_1)$ carefully. Then calculate γ from Eqns. (3.7) and (3.8).

3.4 Forced, Damped, Harmonic Oscillator and Resonance

Set the turntable in motion. The record of $x(t)$ will now be a rather irregular motion (the transient), followed by a steady-state motion of the form $x = A\cos(\omega t + \psi)$. Determine the frequency ω. The steady-state amplitude and phase angle are given by

$$A = \frac{F_0}{m} \frac{1}{\sqrt{(\omega^2 - \omega_0^2)^2 + (\gamma\omega)^2}}. \tag{3.9}$$

$$\tan\psi = \frac{-\gamma\omega}{\omega_0^2 - \omega^2}. \tag{3.10}$$

(!) **Exercise #5.**
Determine A (from the scale attached to the Linear Air Track) for a large number of ω values, and plot a graph (the resonance curve) of A vs. ω. Fit to Equation (3.9) to determine F_0/m, ω, and γ.

Show theoretically that:

1. $A_0 = A(\omega \to 0) = F_0/k_{\text{eff}}$.
2. A is a maximum for $\omega = \omega_r$, with

$$\omega_r^2 = \omega_0^2 - \frac{1}{2}\gamma^2, \tag{3.11}$$

and

$$A_r = A(\omega = \omega_r) = \frac{F_0}{m} \cdot \frac{1}{\gamma\omega_r} \cong \frac{F_0}{m}\frac{1}{\gamma\omega_0} = \frac{A_0\omega_0}{\gamma}. \tag{3.12}$$

Determine A_0, A_r and ω_r from the resonance curve and determine F_0 and γ from them. Compare this γ value with the values determined in Section 3.2.

Two final comments:

1. Note qualitatively the behaviour of the phase angle ψ when ω increases from a value less than ω_r to a value larger than ω_r.

2. The behaviour of a damped oscillator is determined by ω_0 and γ, which are often combined in a "quality factor" $Q = \omega_0/\gamma$.

- Calculate Q for this oscillator. Q determines the "shape" of the resonance curve as follows: Let the two frequencies for which

$$A = \frac{1}{\sqrt{2}} A_r$$

be ω_+ and ω_-. Then the half width of the resonance curve, defined as $\omega_+ - \omega_- = \Delta\omega$, is approximately given by $\Delta\omega \cong \gamma$. Compare the theoretical half width of your resonance curve with the experimental value.

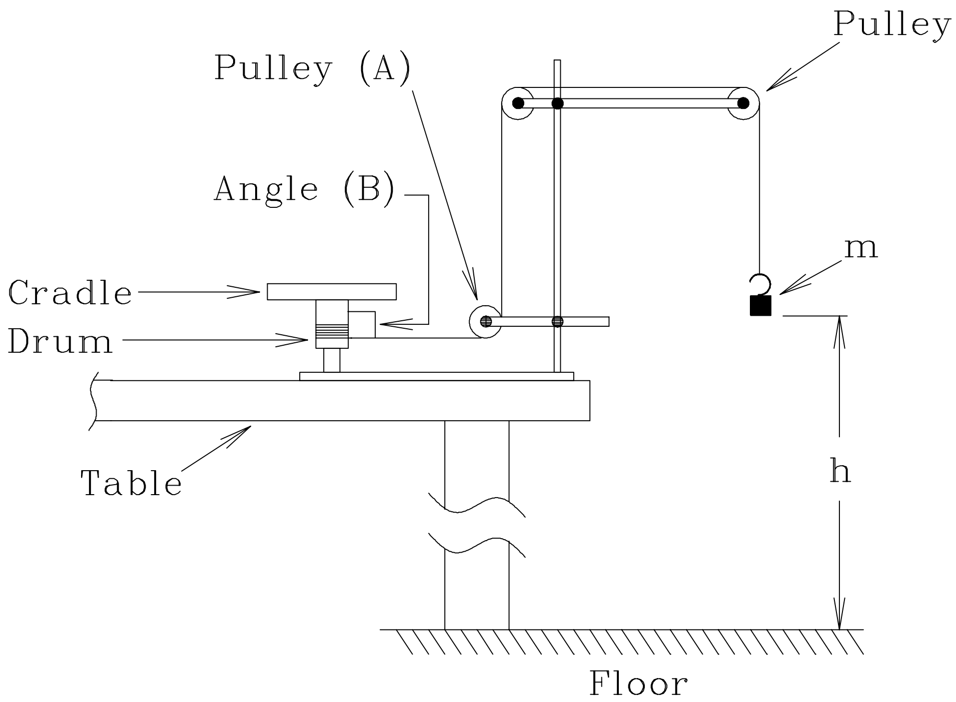
Reference

1. D. Kleppner and R.J. Kolenkow, *An Introduction to Mechanics.* McGraw Hill, 1973. Chapter 10.

Experiment 4

Angular Motion

Before starting the experiment, you need to be familiar with the concept of angular position θ, angular velocity ω, angular acceleration α, torque τ, moment of inertia I. See Kleppner and Kolenkow, Chapter 6.



The equipment consists of a plastic disc bolted to a metal drum (this combination is called the cradle), which can rotate nearly friction free, around a vertical axis. A torque τ can be applied to the cradle with a string-and-pulley combination, at the end of which a mass m can be attached. We arrange the pulleys so that the string leaving the drum is perpendicular to the axis of rotation. Then the torque exerted on the cradle by the string is

$$\vec{\tau} = \vec{r} \times \vec{T}, \tag{4.1}$$

where $\vec{r}$ is the radius of the pulley and $\vec{T}$ is the tension in the string. If the mass m at the other end of the string is falling, starting from rest, over a distance h in a time t, its acceleration is

$$|a| = g - \frac{T}{m} = \frac{2h}{t^2}. \tag{4.2}$$

The linear acceleration a of m is related to the angular acceleration α of the cradle by

$$|a| = |r|\,|\alpha|. \tag{4.3}$$

α is, in turn, related to the moment of inertia of the cradle I_c by

$$\vec{\tau} = I_c\,\vec{\alpha}. \tag{4.4}$$

(!) **Exercise #1.**
Derive Equations (4.1)–(4.4), and combine these equations to show that

$$I_c = mr^2\left[\left(\frac{gt^2}{2h}\right) - 1\right]. \tag{4.5}$$

Using Equations (4.4) and (4.5) to rearrange Equation (4.3) gives

$$\alpha = \frac{a}{r} = \frac{\tau}{I} = \frac{Tr}{I}. \tag{4.6}$$

By combining Equation (4.2) with the result of (4.6), the rotational inertia can be expressed as

$$I = mr^2\left(\frac{g}{a} - 1\right). \tag{4.7}$$

All exercises will be completed with the string wound around the second smallest pulley (r=0.02282 m) on the cradle and the height h set to approximately 1 m.

(!) **Exercise #2.**
Using the PhysicaLab software, select the Dig2 photogate input channel and collect 20 points of data, with a timing of 0.5 seconds between points. To determine the acceleration, fit the data to A+B*x+C*x**2/2, and record the values for the acceleration of each run. Each weight will be run twice, and an average will be calculated. You will now evaluate the rotational inertia I_c of the unloaded cradle using Equation (4.7). Using the equation on its own would not lead to the correct I_c value. This is because the mass required to overcome the friction of the pulley system, m_0, is not taken into account. There is a method that does not require m_0 to be known. This method starts with a rearrangement of Equation (4.7), so that m is expressed as a function of a:

m (kg)	0.020	0.030	0.040	0.050
a_1 (m/s^2)				
a_2 (m/s^2)				
$\langle a\rangle$ (m/s^2)				

Table 4.1: Acceleration data for unloaded cradle assembly

$$I_c = m_T r^2\left(\frac{g}{a} - 1\right) \approx m_T r^2\left(\frac{g}{a}\right) = (m - m_0)\left(\frac{gr^2}{a}\right) \rightarrow m = m_0 + \left(\frac{I_c}{gr^2}\right)a. \tag{4.8}$$

The equation is simplified by the fact that $a \ll g$, and by taking into account that the mass offset m_0 and the mass m used make up the total mass $m_T = m - m_0$ that causes the tension on the string. The resulting equation is that of a straight line with slope $I_c/(gr^2)$ and y-intercept equal to m_0.

Using PHYSICA, make a plot of m vs. $\langle a\rangle$, fit to a straight line, and determine I_c.

(!) **Exercise #3. Rotational inertia of the cradle and disc.**
Place the disc on the cradle and repeat the measurements to obtain the moment of inertia I_t of the cradle-plus-disc. Subtract I_c from I_t to obtain I_{disc}, the moment of inertia of the disc alone. Compare the measured value of I_{disc} with the theoretical value which you can calculate from the disc's weight and dimensions.

m (kg)	0.020	0.030	0.040	0.050
a_1 (m/s^2)				
a_2 (m/s^2)				
$\langle a \rangle$ (m/s^2)				

Table 4.2: Acceleration data for cradle-and-disc assembly

(!) **Exercise #4. Rotational inertia of the cradle and ring.**
Replace the disc with the ring on the cradle and repeat the measurements to obtain the moment of inertia I_t of the cradle-plus-ring. Subtract I_c from I_t to obtain I_{ring}, the moment of inertia of the ring alone. Compare the measured value of I_{ring} with the theoretical value which you can calculate from the ring's weight and dimensions.

m (kg)	0.020	0.030	0.040	0.050
a_1 (m/s^2)				
a_2 (m/s^2)				
$\langle a \rangle$ (m/s^2)				

Table 4.3: Acceleration data for cradle-and-ring assembly

(!) **Exercise #5. Rotational inertia of the cradle and cube.**
Replace the ring with the cube on the cradle and repeat the measurements to obtain the moment of inertia I_t of the cradle-plus-cube. Subtract I_c from I_t to obtain I_{cube}, the moment of inertia of the cube alone. Compare the measured value of I_{cube} with the theoretical value which you can calculate from the cube's weight and dimensions.

m (kg)	0.020	0.030	0.040	0.050
a_1 (m/s^2)				
a_2 (m/s^2)				
$\langle a \rangle$ (m/s^2)				

Table 4.4: Acceleration data for cradle-and-cube assembly

(!) **Exercise #6. Rotational inertia of the cradle and sphere.**
Replace the cube with the sphere on the cradle and repeat the measurements to obtain the moment of inertia I_t of the cradle-plus-sphere. Subtract I_c from I_t to obtain I_{sphere}, the moment of inertia of

the sphere alone. Compare the measured value of I_{sphere} with the theoretical value which you can calculate from the sphere's weight and dimensions.

m (kg)	0.020	0.030	0.040	0.050
a_1 (m/s^2)				
a_2 (m/s^2)				
$\langle a \rangle$ (m/s^2)				

Table 4.5: Acceleration data for cradle-and-sphere assembly

(!) **Exercise #7. Rotational inertia of the cradle and cone.**
Replace the sphere with the cone on the cradle and repeat the measurements to obtain the moment of inertia I_t of the cradle-plus-cone. Subtract I_c from I_t to obtain I_{cone}, the moment of inertia of the cone alone. Compare the measured value of I_{cone} with the theoretical value which you can calculate from the cone's weight and dimensions.

m (kg)	0.020	0.030	0.040	0.050
a_1 (m/s^2)				
a_2 (m/s^2)				
$\langle a \rangle$ (m/s^2)				

Table 4.6: Acceleration data for cradle-and-cone assembly

(!) **Exercise #8. Rotational inertia of the cradle and pyramid.**
Replace the cone with the pyramid on the cradle and repeat the measurements to obtain the moment of inertia I_t of the cradle-plus-pyramid. Subtract I_c from I_t to obtain $I_{pyramid}$, the moment of inertia of the pyramid alone. Compare the measured value of $I_{pyramid}$ with the theoretical value which you can calculate from the pyramid's weight and dimensions.

m (kg)	0.020	0.030	0.040	0.050
a_1 (m/s^2)				
a_2 (m/s^2)				
$\langle a \rangle$ (m/s^2)				

Table 4.7: Acceleration data for cradle-and-pyramid assembly

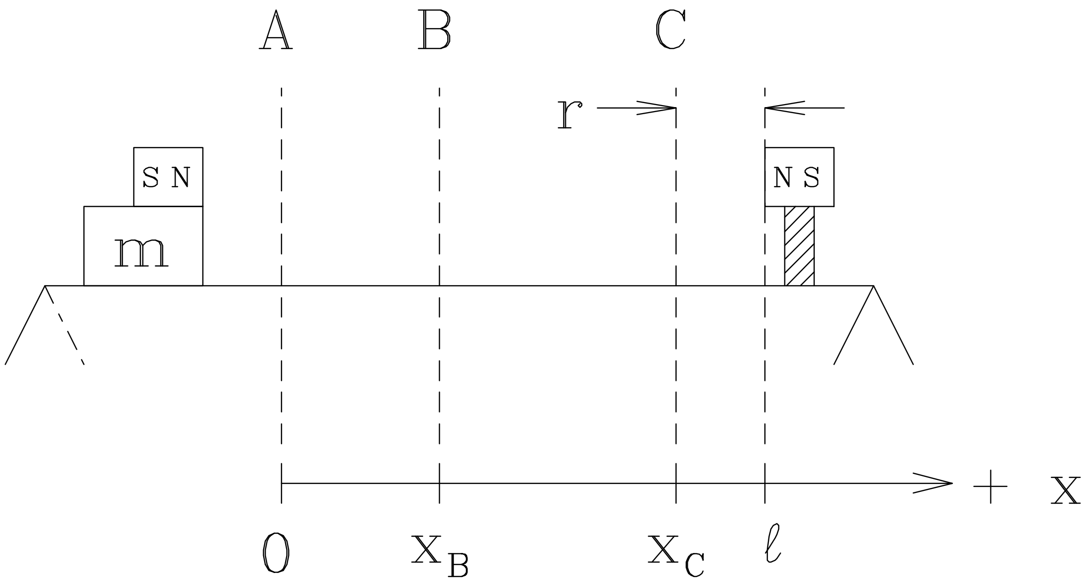
Experiment 5

Determination of a Potential Energy Curve on a Linear Air Track

5.1 Theory

A glider of mass m is moving with a velocity $\vec{v}_0$ towards a stationary magnet. The force between the two magnets is repulsive, and at point C the glider will reverse its direction of motion. The position of C depends on m, v_0 and the strength of the magnetic interaction. This strength is given by a potential function $U(r)$, with r as the separation of the two magnets. If the track was truly free of friction, then $U(r)$ is easy to determine: measure v_0 at a position, say at A, where $U(r)$ is negligibly small. If the turning point C is at position x_c, then conservation of mechanical energy gives (where $r = \ell - x_c$):

"Total Energy at A" (i.e. $x = 0$) equals "Total Energy at C" (i.e. $x = x_c$).



$$\frac{1}{2}mv_0^2 + 0 = 0 = U(r). \tag{5.1}$$

Regrettably, the track is not exactly friction-free, and the frictional loss in kinetic energy while the glider travels from A to C is not a negligible quantity. We can take the friction losses into account as follows: Apply the Work-Energy theorem to the motion from A to C:

$$W_m + W_f = \left(\frac{1}{2}mv^2\right) \text{ (at C)} - \left(\frac{1}{2}mv^2\right) \text{ (at A)}, \tag{5.2}$$

where W_m is the work done by the magnetic forces and W_f is the work done by the friction forces. Since the magnetic force is zero at A, we have

$$W_m = -U(x_r). \tag{5.3}$$

The right-hand-side of Equation (5.2) equals $-\frac{1}{2}mv_0^2$, where v_0 is the speed at $A(x = 0)$.

5.1.1 Calculation of the Term W_f

The origin of the frictional force F_f is the friction in the air layer between the glider and the track. For small glider velocities v this force is proportional to v, producing an acceleration $a = -kv$, where k is a constant. For this type of acceleration, we know that velocity v and position x depend on time t as follows:

$$a = -kv \tag{5.4}$$

$$v = v_0 \exp(-kt) \tag{5.5}$$

$$x = \left(\frac{v_0}{k}\right)(1 - \exp(-kt)). \tag{5.6}$$

Therefore the relations between a, v and x are:

$$a = k^2x - kv_0 \tag{5.7}$$

$$v = v_0 - kx. \tag{5.8}$$

Therefore, the work done by the frictional force while the glider moves from A to C is approximately[1] given by

$$W_f = \int_0^{x_r} \vec{F}_f \cdot d\vec{x} = m\int_0^{x_r} (k^2x - kv_0)\, dx = m\left(\frac{1}{2}k^2x_r^2 - kv_0x_r\right). \tag{5.9}$$

Combining Equations (5.2), (5.3), and (5.9) gives $U(x_r)$:

$$U(x_r) = \frac{1}{2}mv_0^2 + \frac{1}{2}mk^2x_r^2 - mkv_0x_r. \tag{5.10}$$

We can determine k by measuring the velocity v_B of the glider at a point B (coordinate x_B) where the magnetic force is still negligible; from Equation (5.5) we have

$$k = \left(\frac{v_0 - v_B}{x_B}\right). \tag{5.11}$$

Therefore, all quantities in Equation (5.10) can be measured, and $U(x_r)$ can be determined.

5.2 Procedure

Carefully level the air track; position the fixed magnet and the two photogates. Set $x_B \simeq 0.3$ m and $\ell \simeq 0.6$ m. The photogates at A and B measure the velocities of the glider by recording the times it takes the glider to interrupt a light beam. Let these times be Δt_0 (at A) and Δt_B (at B); then $v_0 = \ell^*/\Delta t_0$ and $v_B = \ell^*/\Delta t_B$, where ℓ^* is the length of the glider. First determine carefully the quantities: m, x_B, ℓ and ℓ^*. Then give the glider a velocity in the range of $0.1 - 0.2$ ms^{-1}, and determine, for a given run, $\Delta t_0, \Delta t_B$ and x_c. Do a number of runs, so that you can calculate $U(x_r)$ for a range of x_r values. Arrange the data in tabular form.

[1] The result is not exact, since close to C, where the magnetic force becomes important, the acceleration $\vec{a}$ is not proportional to $-\vec{v}$ anymore.

5.3 Data Analysis

Do all calculations in the MKS system of units; record all calculated results in tabular form. For each run:

1. Calculate v_0 and v_B from $\ell^*, \Delta t_0$ and Δt_B.

2. Calculate k, using Equation (5.11), with v_0, v_B, and x_B.

3. From the values v_0, m, k and x_c for each run, calculate each term in Equation (5.10), and sum them to obtain $U(r)$. (You really obtain first $U(x_c)$), but you can convert the x_c values into r values using $r = \ell - x_c$.

4. So now you have $U(r)$ values for different r values. Can you now "guess" at the mathematical relation between $U(r)$ and r? For example: Is $U(r)$ linear in r, that is: $U(r) = ar + b$? You can check this by plotting $U(r)$ vs. r on linear graph paper, and see if your data points fall on a straight line. If they do, you can calculate a and b. If they do not, then plot $U(r)$ vs. r on full logarithmic graph paper. If the data points fall on a straight line now, then you know that $U(r) = Kr^n$, and from your graph you can determine K and n. If this does not work, plot your data on semi-logarithmic graph paper, to see if they obey $U(r) = A\exp(Br)$.

In your report, derive Equations (5.7) and (5.8).

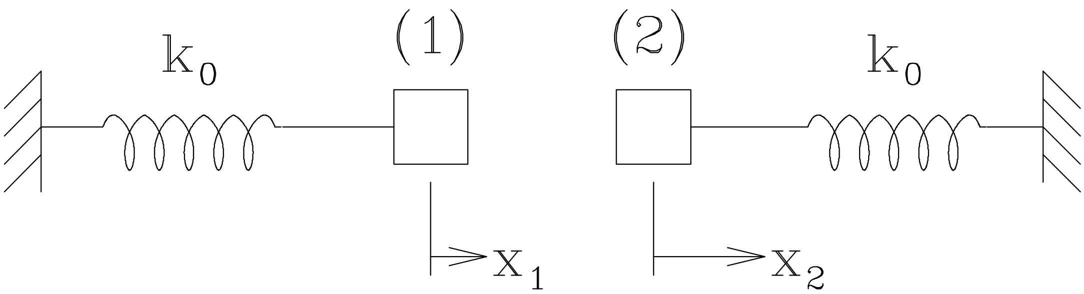
Experiment 6

Coupled Oscillators

In this experiment you will examine the behaviour of coupled pendula, and investigate the dependence of the normal mode frequencies on the strength of the coupling. Before starting the experiment you have to be familiar with the concepts of normal modes, exchange or beat frequencies, and the theory of the simple physical pendulum.

6.1 Coupled Pendula

First, consider two identical oscillators, labelled 1 and 2.



Their positions are given by the coordinates $x_1(t)$ and $x_2(t)$; the x_1 and x_2 coordinates are measured from the relaxed positions. We have the two differential equations,

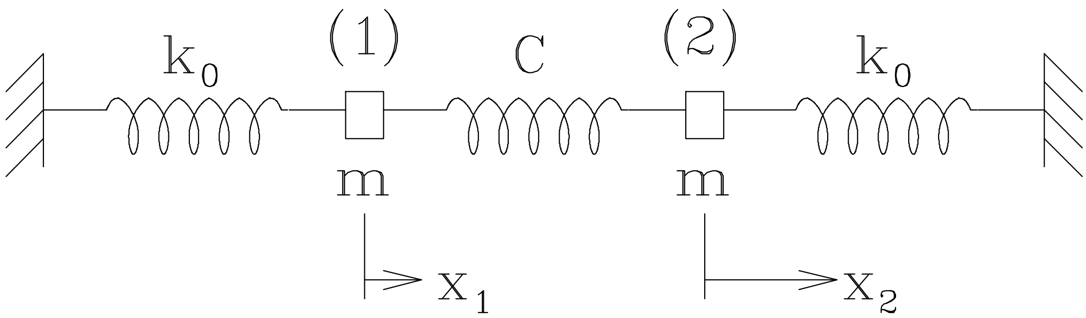
$$\ddot{x}_1 + \omega_0^2 x_1 = 0, \quad (6.1)$$
$$\ddot{x}_2 + \omega_0^2 x_2 = 0, \quad (6.2)$$

with the solutions:

$$x_1 = A_1 \cos(\omega_0 t + \psi_1), \quad (6.3)$$
$$x_2 = A_2 \cos(\omega_0 t + \psi_2), \quad (6.4)$$

and identical frequencies $\omega_0 = \sqrt{k_0/m}$ for the two oscillators.

Now we couple the two oscillators with a coupling spring, constant C:



The motion of oscillator 1 now directly influences the motion of oscillator 2, via the extensions and compressions of the coupling spring. The differential equations of motion are now:

$$\ddot{x}_1 + \omega_0^2 x_1 + k\,(x_1 - x_2) = 0, \tag{6.5}$$
$$\ddot{x}_2 + \omega_0^2 x_2 - k\,(x_1 - x_2) = 0, \tag{6.6}$$

where $k = C/m$.

(!) **Exercise #1.**
Derive Equations (6.5) and (6.6).

The solutions to Equations (6.5) and (6.6) are:

$$x_1 = \frac{1}{2}A_0\cos(\omega_0 t + \psi_0) + \frac{1}{2}A_1\cos(\omega_1 t + \psi_1), \tag{6.7}$$
$$x_2 = \frac{1}{2}A_0\cos(\omega_0 t + \psi_0) - \frac{1}{2}A_1\cos(\omega_1 t + \psi_1), \tag{6.8}$$

where $\omega_0^2 = k/m$ and $\omega_1^2 = \omega_0^2 + 2k$.

The motion of the oscillators is now the superposition of two simple harmonic motions, with the (different) frequencies ω_0 and ω_1. The constants A_0, A_1, ψ_0 and ψ_1 depend on initial conditions.

The motions are particularly simple for the following cases:

1. Initial conditions

$$\left.\begin{array}{ll} \dot{x}_1 = 0, & \dot{x}_2 = 0 \\ x_1 = A, & x_2 = A \end{array}\right\} \text{at } t = 0.$$

Then

$$x_1 = x_2 = A\cos\omega_0 t. \tag{6.9}$$

Both oscillators vibrate in phase at the same frequency ω_0; this motion is called the "first normal mode of vibration" for our system.

2. Initial conditions

$$\left.\begin{array}{ll} \dot{x}_1 = 0, & \dot{x}_2 = 0 \\ x_1 = A, & x_2 = -A \end{array}\right\} \text{at } t = 0.$$

Then

$$x_1 = -x_2 = A\cos\omega_1 t. \tag{6.10}$$

This is the second normal mode.

3. Initial conditions

$$\left.\begin{array}{ll} \dot{x}_1 = 0, & \dot{x}_2 = 0 \\ x_1 = 0, & x_2 = A \end{array}\right\} \text{at } t = 0.$$

Then

$$x_1 = A\sin\frac{1}{2}(\omega_1 - \omega_0)t \cdot \sin\frac{1}{2}(\omega_1 + \omega_0)t, \quad (6.11)$$

$$x_2 = A\cos\frac{1}{2}(\omega_1 - \omega_0)t \cdot \cos\frac{1}{2}(\omega_1 + \omega_0)t. \quad (6.12)$$

(!) **Exercise #2.**
Derive Equations (6.11) and (6.12) from (6.7) and (6.8) with the specified initial conditions.

Equation (6.11) says that the motion of oscillator 1 is a harmonic motion (with frequency $\frac{1}{2}(\omega_1+\omega_0)$) with an amplitude $A\sin\frac{1}{2}(\omega_1-\omega_0)t$ which changes with time; in fact, the amplitude itself is oscillating with a frequency $\frac{1}{2}(\omega_1-\omega_0)$. So at time $t = 0$ the amplitude is zero; at $t = \pi/(\omega_1-\omega_0)$ the amplitude is A; at $t = 2\pi/(\omega_1-\omega_0)$ the amplitude is zero again and at $t = 3\pi/(\omega_1-\omega_0)$ the amplitude is $-A$, *etc.*

(!) **Exercise #3.**
Show that the amplitude of oscillator 2 is equal to $\pm A$ when the amplitude of oscillator 1 is zero, and vice versa.

The two oscillators periodically exchange their kinetic energy; the exchange period is $T_e = 2\pi/(\omega_1-\omega_0)$ and the exchange frequency $\omega_e = (\omega_1-\omega_0)$. That is: if at $t = 0$ oscillator 2 has all the kinetic energy (and oscillator 1 is at rest momentarily), then at time $t = \pi/(\omega_1-\omega_0)$ all the kinetic energy is in oscillator 1, (and oscillator 2 is at rest). The "exchange period" is therefore: $T_e = 2\pi(\omega_1-\omega_0)$ and the exchange frequency: $\omega_e = (\omega_1-\omega_0)$.

6.2 Single Pendulum

We will now consider the oscillations of one pendulum. Our experimental pendulum is a simple rod, length L, total mass M, oscillating around one end. When the pendulum is given a small angular displacement α from the vertical, a torque J acts in it.

$$J = -Mg\,d\sin\alpha, \quad (6.13)$$

where g is the acceleration of gravity, and d is the distance from the centre of gravity to the axis of oscillation. Then the equation of motion of the pendulum is

$$I\frac{d^2\alpha}{dt^2} = -Mg\,d\sin\alpha, \quad (6.14)$$

where I is the moment of inertia with respect to the axis of oscillation. For our pendulum, $I \cong \frac{1}{3}ML^2$ and $d \cong \frac{1}{2}L$. If the angle of oscillation α is small we may approximate $\sin\alpha \cong \alpha$, and Equation (6.2) becomes

$$\frac{d^2\alpha}{dt^2} + \omega_0^2\,\alpha = 0, \quad (6.15)$$

where $\omega_0^2 = Mgd/I$. We see from Equation (6.2) that the pendulum will execute harmonic oscillations with respect to the vertical with a frequency ω_0.

6.3 Two Coupled Pendula

Now consider the situation when we couple the motions of two identical pendula by means of a spring with spring constant K. Let the relaxed spring length be equal to the distance between the axes of oscillation, and let the spring be connected to the pendulum rods at a distance h from the axes of oscillation (see Figure 6.1). Then the differential equation of motion of pendulum 1 is:

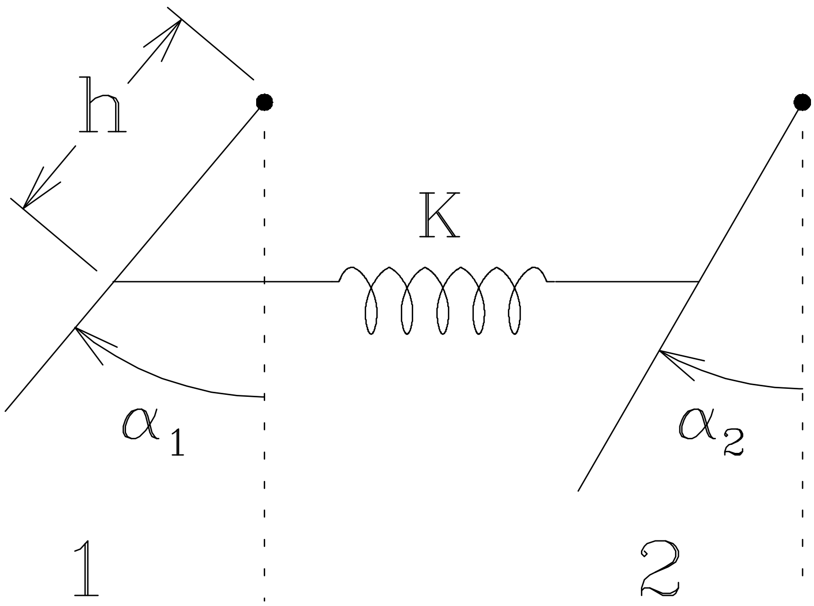


Figure 6.1: Two coupled pendula

$$I\frac{d^2\alpha_1}{dt^2} = -Mg\,d\sin\,\alpha_1 - K(h\sin\alpha_1 - h\sin\alpha_2)\cdot h\cos\,\alpha_1. \tag{6.16}$$

For small α this can be written as

$$\frac{d^2\alpha_1}{dt^2} = -\frac{Mgd}{I}\alpha_1 - \frac{Kh^2}{I}\alpha_1 + \frac{Kh^2}{I}\alpha_2. \tag{6.17}$$

For pendulum 2 the equation of motion is

$$\frac{d^2\alpha_2}{dt^2} = -\frac{Mgd}{I}\alpha_2 - \frac{Kh^2}{I}\alpha_2 + \frac{Kh^2}{I}\alpha_1. \tag{6.18}$$

(!) **Exercise #4.**
Derive Equation (6.17) by considering the torques on the pendulum 1. (Hint: torques are exerted on 1 by its weight and by the extended spring K.)

Equations (6.17) and (6.18) are identical in form to Equations (6.5) and (6.6) of the general theory, and the theory can now be applied in a straightforward way to our system of two coupled pendula.

(!) **Exercise # 5.**
Carefully compare Equations (6.5) and (6.6) with Equations (6.17) and (6.18). Find expressions for ω_0, ω_1 and k from the general theory in terms of I, K, M, d and h.

Procedure

1. Remove the coupling spring from two adjacent pendula. For each pendulum, measure and calculate its natural frequency ω_0. If necessary, tune the two pendula by adjusting the tuning screw at the bottom-end of the pendulum rod, so that the uncoupled pendula have identical frequencies.

2. Couple the two pendula with the spring at a distance $h = 10$ cm. Measure the frequencies ω_0 and ω_1 of the two normal modes, and the exchange or beat frequency ω_e.

3. Increase h in steps of 10 cm, up to $h = 70$ cm and for each h measure ω_0, ω_1 and ω_e.

4. According to the theory, you should find:

   - $\omega_e = (\omega_1 - \omega_0)$
   - ω_1 proportional to h^2.

Do your data obey these relations?

Experiment 7

Kater's Pendulum

The period T of a physical pendulum is given by

$$T = 2\pi\sqrt{\frac{I}{Mgd}}, \tag{7.1}$$

where

I: Moment of Inertia with respect to the axis of oscillation (AO).
M: Total mass of the pendulum.
g: Acceleration due to gravity.
d: Distance between the centre of mass (CM) of the pendulum and the AO.

When the amplitude of oscillation is not small, so that the approximation $\sin\theta \cong \theta$ ceases to be valid, then T is given by

$$T = 2\pi\sqrt{\frac{I}{Mgd}}\left[1 + \frac{1}{2^2}\cdot\sin^2\left(\frac{\theta_m}{2}\right) + \frac{1}{2^2}\cdot\frac{3^2}{4^2}\cdot\sin^4\left(\frac{\theta_m}{2}\right) + \cdots\right], \tag{7.2}$$

where θ_m is the maximum angular displacement.

Kater's pendulum consists of a bar with two fixed knife edges, AO_1 and AO_2; two different masses M_1 and M_2 can be clamped to the bar. The positions of M_1 and M_2 along the bar are adjusted until the period of oscillations T_1 around AO_1 is equal to the period of oscillation T_2 around AO_2. In this condition we then have

$$T_1 = T_2 = T_0 = \sqrt{\frac{4\pi^2\ell_0}{g}}, \tag{7.3}$$

where ℓ_0 is the distance between the knife edges.

7.1 Procedure

Set up Kater's pendulum, making sure it swings freely. Fasten (and fix) the small mass M_1 to the bar, outside AO_1 and AO_2, and measure its distance ℓ to AO_1. Swing the pendulum, with small θ_m, with AO_1 as the pivot point, and measure T_1 a few times with the photogate. Be sure that the pendulum swings "straight" through the gate.

Now reverse the pendulum; swing it around AO_2 and measure T_2. Unless you are very lucky, T_1 and T_2 will be appreciably different. Change the position of M_2 (that is, change ℓ), and measure T_1 and T_2 again. Repeat this procedure a few times and plot curves of T_1 vs. ℓ and T_2 vs. ℓ on one graph. The intersection of these two curves indicates the value of ℓ for which T_1 is approximately equal to T_2. Guided by this graph, change ℓ by small amounts in the right direction until you experimentally find $T_1 = T_2 = T_0$ within experimental error.

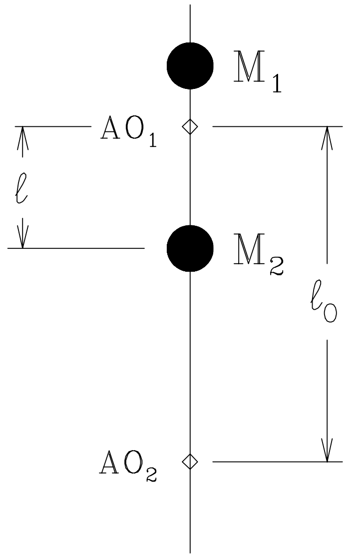


Figure 7.1: Dimensions of Kater's Pendulum

For our pendulum, $\ell_0 = 99.48 \pm 0.01$ cm; with your T_0 value you can now calculate g from Equation (7.3). There are in fact *two* positions of M_2 for which $T_1 = T_2 = T_0$; find both positions, and compare the two values of g, and their error.

The inherent accuracy of this experiment is quite high. Give a complete error analysis, including a discussion of whether your measured T values should be corrected for large oscillation amplitudes θ_m. Use Equation (7.2) to prove this.

References

1. D. Kleppner and R.J. Kolenkow, *An Introduction to Mechanics.* McGraw Hill, 1973. Chapter 6.

2. N. Feather, *An Introduction to the Physics of Mass, Length and Time.* pp. 187–194.

Appendix A

Vernier Scales

Certain measurements require more precision than can be obtained using common measuring devices, e.g. a metre stick for linear quantities, or a protractor for angular quantities. In this laboratory, you will have occasion to use a *vernier scale* to take measurements to a greater precision than is possible with these devices.

Vernier Calipers

Vernier calipers are used to make linear measurements to a precision on the order of 0.001 mm.

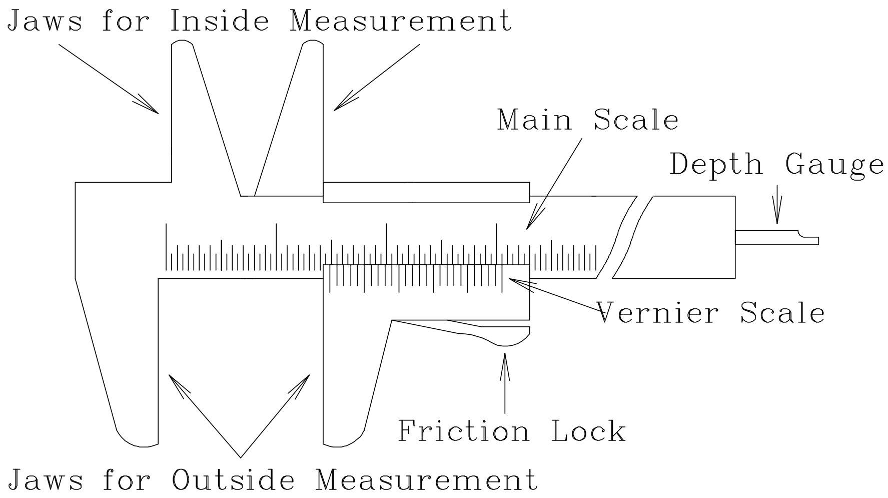


Figure A.1: The Vernier Caliper

Operation of Vernier Calipers

1. Zero Error: Before taking any measurements close the calipers and check for any zero error. If an error exists, record it and adjust any future measurements.

2. Care in Operation: The calipers must be closed gently on the object. Forcing the calipers will result in an error in measurement. They can be closed properly by applying pressure via the friction lock.

3. Cross-Cornered Measurement: It is necessary to check for every measurement that the calipers are perpendicular to the object being measured.

Main and Vernier Scales

The main scale is divided into 0.05 cm (0.5 mm) units, and the vernier scale is divided into 0.002 cm (0.02 mm) units.

Reading the Scales

1. Locate the "0" line on the vernier scale, and note which main scale division it is immediately after, e.g. 7.150 cm on the main scale in Figure A.2.

2. Scan along the line where the main and vernier scales meet, and note which *one* vernier scale division is directly in line with a main scale division, e.g. 0.014 cm on the vernier scale in Figure A.2.

3. Add the main and vernier scale readings to obtain the final reading. In Figure A.2 for example, 7.150 cm + 0.014 cm = 7.164 cm.

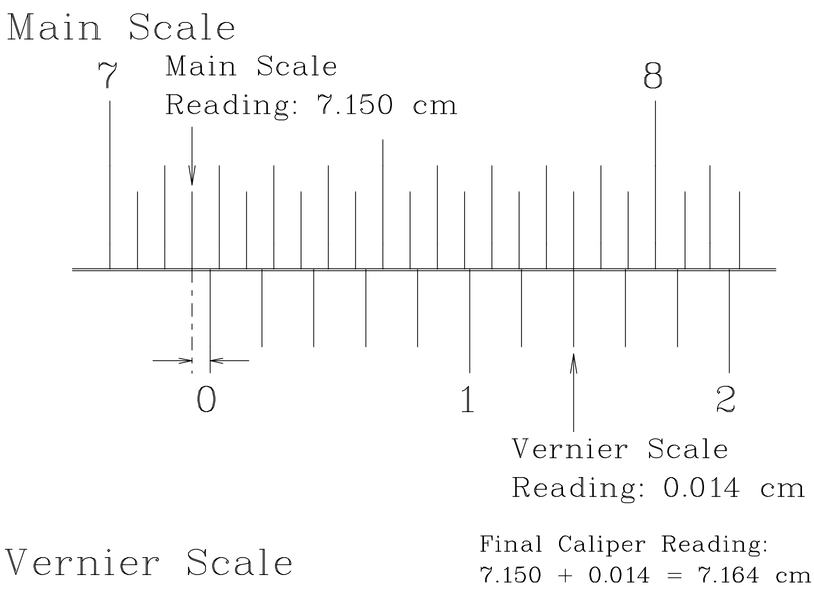


Figure A.2: Example of Vernier Caliper Reading – 7.164 cm

The Angular Vernier

Units of Angular Measure

Your angular measurements will consist of two parts: degrees, and a fraction of a degree called "minutes" (symbol $'$). One degree is divided into sixty minutes, and one minute is divided into sixty seconds (symbol $''$), although the instrument that you are using has a precision of $1'$.

Main and Vernier Scales

As with the calipers, two scales are used. The main scale is circular, divided into 720 half-degree units, permitting measurements from this scale to a precision of 0.5°. The vernier scale allows a precision of $1'$ to be obtained.

Reading the Scales

1. Locate the "0" line on the vernier scale, and note which main scale division it is immediately after, e.g. $212°30'$ on the main scale in Figure A.3. Note that the numbers on the main and vernier scales increase from right to left, and not from left to right as you are used to reading.

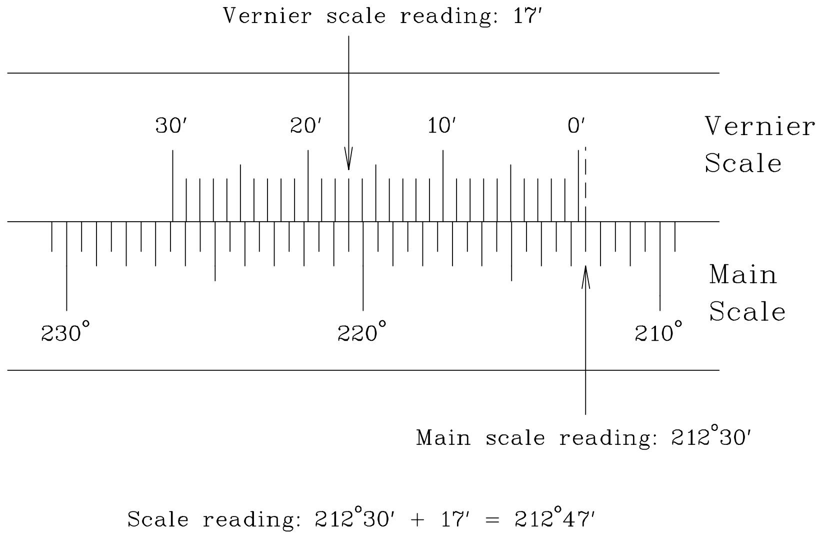


Figure A.3: Example of Angular Vernier Scale Reading – $212°47'$

2. Scan along the line where the main and vernier scales meet, and note which *one* vernier scale division is directly in line with a main scale division, e.g. $17'$ on the vernier scale in Figure A.3.

3. Add the main and vernier scale readings to obtain the angular scale reading, e.g. $212°30' + 17' = 212°47'$ in Figure A.3.

Appendix B

Graphing

Title

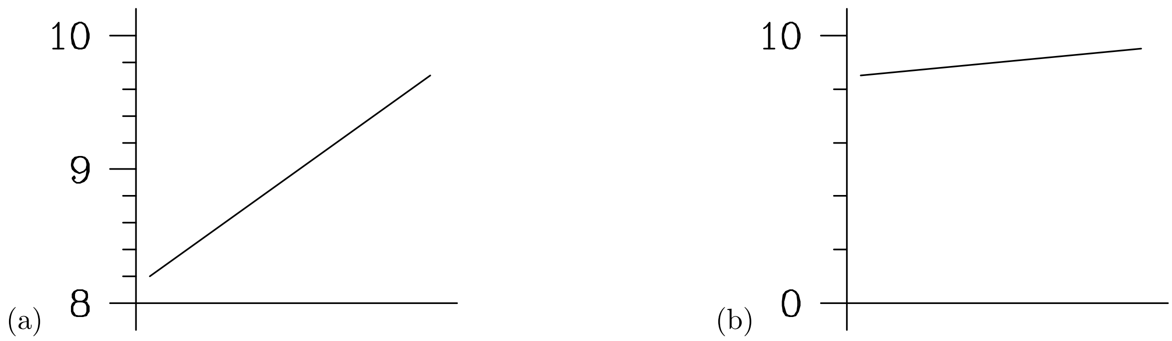
Each graph must include a descriptive title that leaves no confusion as to what is being plotted.

Axes

Each axis must be clearly labelled (including proper units) indicating what is plotted.

Scales

Axis scales should be chosen so that the graph is as large as possible. Remember that scales *do not* have to start at zero (although they may, if it is appropriate), especially if doing so would make the graph unreadable, as shown below in (b).



The scales should also be chosen so that they are easily readable. Suppose you are plotting on graph paper with major divisions of 1 cm and minor divisions of 1 mm (0.1 cm). With scale (a) in the following diagram, it is much easier to estimate between divisions (when taking points to calculate a graph slope, for example) than it is with scale (b).

Graphing Convention

In this laboratory and in other courses, you will often be told to plot "'quantity #1' versus 'quantity #2'" The proper way to plot these is with "quantity #1" on the vertical axis and "quantity #2" on the horizontal axis. In other words, you will be instructed to plot "*vertical* versus *horizontal*".

Figure B.1: (a) 1 mm (on paper) : 1 s; (b) 1 mm (on paper) : 0.9 s

Error Bars

When plotting experimental points on a graph, uncertainties are indicated using "error bars". For any experimental points $(x \pm \sigma x, y \pm \sigma y)$, the error bars will consist of a pair of line segments of length $2\sigma x$ and $2\sigma y$, which are parallel to the x and y axes respectively. These line segments are centered on the point (x, y). The correct point should lie within the rectangle formed using the error bars as sides. The rectangle

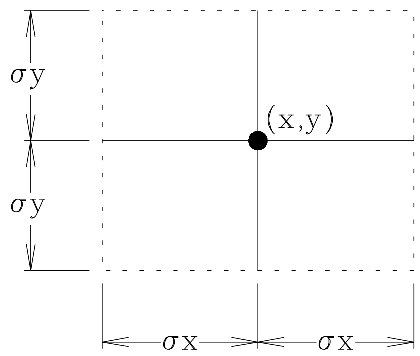


Figure B.2: Error Bars for Point (x, y)

is indicated by the dotted lines in figure B.2. Note that only the error bars, and not the rectangle are drawn on the graph.

Line of Best Fit

To analyze your data (assuming a linear relationship), it is necessary to draw a line of "best fit". This line should approximate your data as much as possible. The best fit line can be determined reasonably well manually (by eye), and the slope determined in the usual graphical manner. There are analytical ways to determine the line of best fit. The advantage of doing this is that all people using the same data will get the same results. One method is to use the "Linear Regression" (LR) function on a calculator or spreadsheet program. Based on the data (i.e. x and y coordinates) given to it, a LR routine will return, among other things, the slope and vertical intercept of the line of best fit.

Be aware that performing a LR analysis on non-linear data will produce meaningless results. The graphs in Figure B illustrate several such cases. These graphs show four different data sets that give identical slopes and vertical intercepts via LR.

You should plot the data points to determine visually if a LR analysis is indeed valid.

Calculating the Slope of a Linear Graph

The slope (m) of a straight-line graph is determined by choosing two points, $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, on the line of best fit (not from the original data) and evaluating equation (B.1). Note that the two points

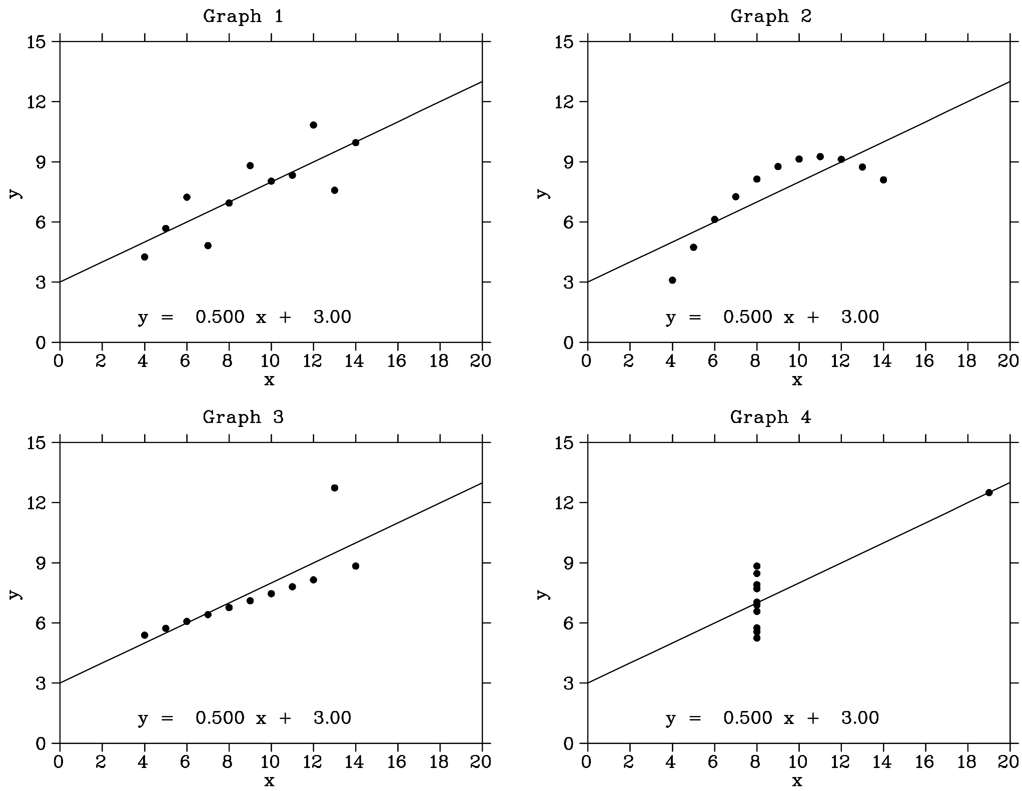
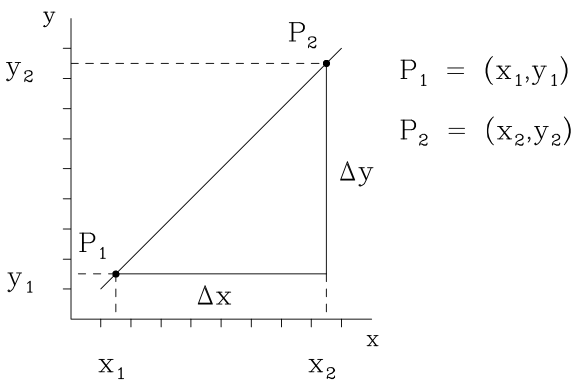


Figure B.3: Four graphs producing identical linear regression results

should be as far apart as possible.



$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{(B.1)}$$

Calculating the Uncertainty of the Slope of a Linear Graph

Consider figure B.4 showing data points for a linear relationship (in this case, the position of an object measured over a period of several seconds). The slope of the line formed by the data points is taken to be the slope of the line which best fits the points (line 2). To determine the uncertainty, lines 1 and 3 are drawn. They give the maximum and minimum slopes, respectively, which satisfy the data points. The uncertainty can be taken as one-half the difference between these two slopes, as in equation (B.2).

$$\sigma(\text{slope}) = \frac{\text{slope}_{\text{max}} - \text{slope}_{\text{min}}}{2} \tag{B.2}$$

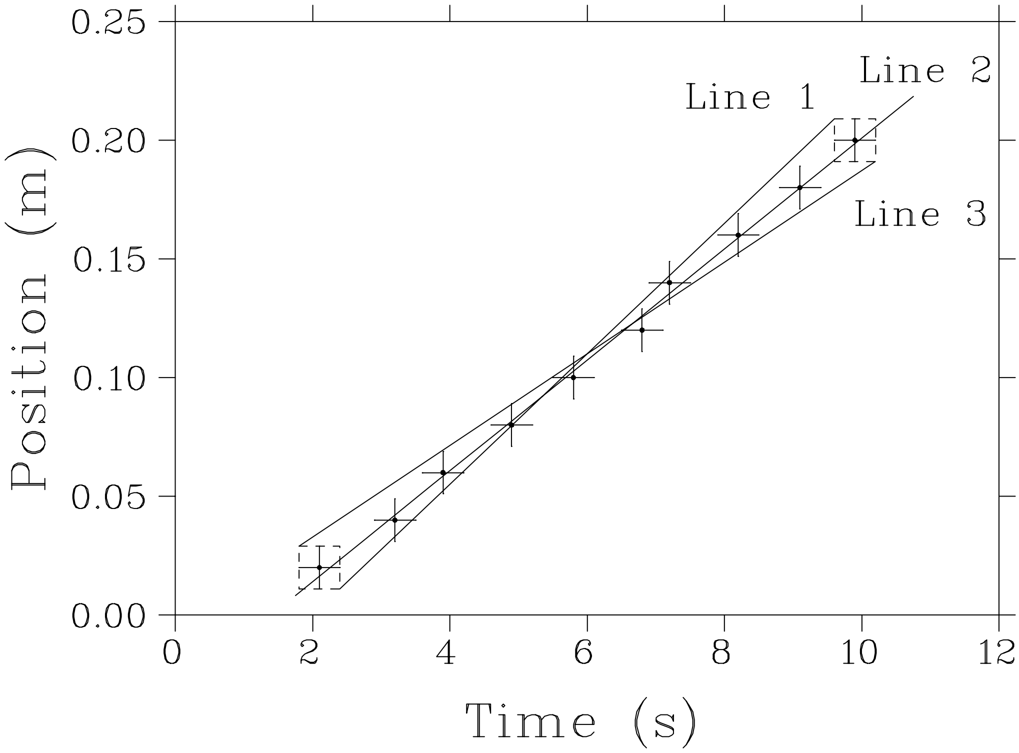


Figure B.4: Lines of maximum slope (line 1), best fit (line 2), minimum slope (line 3)

Line	Slope (or speed) (m/s)
1	0.0275
2	0.0233
3	0.0193

Therefore the slope of the relationship plotted (the speed of the object) in figure B.4 is 0.023 ± 0.004 m/s.