

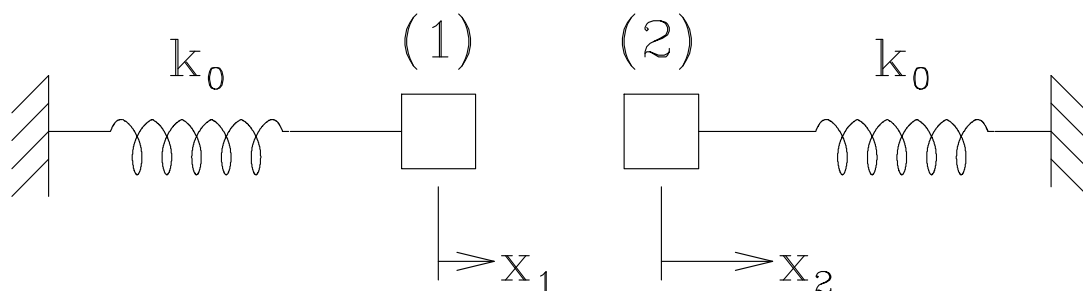
## Experiment 6

# Coupled Oscillators

*In this experiment you will examine the behaviour of coupled pendula, and investigate the dependence of the normal mode frequencies on the strength of the coupling. Before starting the experiment you have to be familiar with the concepts of normal modes, exchange or beat frequencies, and the theory of the simple physical pendulum.*

### 6.1 Coupled Pendula

First, consider two identical oscillators, labelled 1 and 2.



Their positions are given by the coordinates  $x_1(t)$  and  $x_2(t)$ ; the  $x_1$  and  $x_2$  coordinates are measured from the relaxed positions. We have the two differential equations,

$$\ddot{x}_1 + \omega_0^2 x_1 = 0, \quad (6.1)$$

$$\ddot{x}_2 + \omega_0^2 x_2 = 0, \quad (6.2)$$

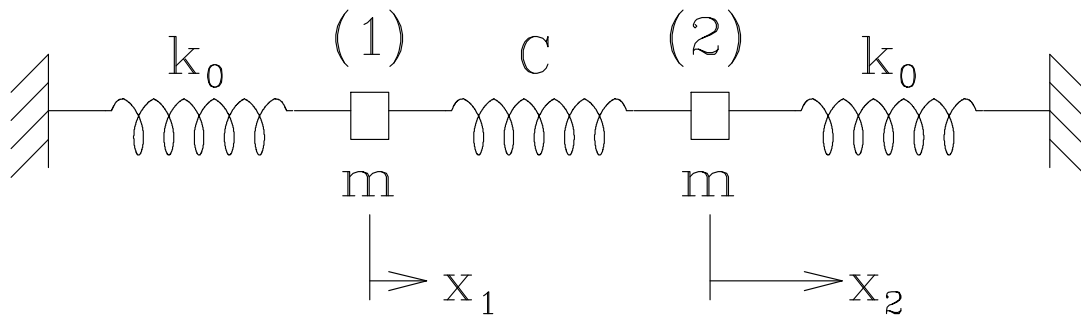
with the solutions:

$$x_1 = A_1 \cos(\omega_0 t + \psi_1), \quad (6.3)$$

$$x_2 = A_2 \cos(\omega_0 t + \psi_2), \quad (6.4)$$

and identical frequencies  $\omega_0 = \sqrt{k_0/m}$  for the two oscillators.

Now we couple the two oscillators with a coupling spring, constant C:



The motion of oscillator 1 now directly influences the motion of oscillator 2, via the extensions and compressions of the coupling spring. The differential equations of motion are now:

$$\ddot{x}_1 + \omega_0^2 x_1 + k(x_1 - x_2) = 0, \quad (6.5)$$

$$\ddot{x}_2 + \omega_0^2 x_2 - k(x_1 - x_2) = 0, \quad (6.6)$$

where  $k = C/m$ .

ⓘ **Exercise #1.**

Derive Equations (6.5) and (6.6).

The solutions to Equations (6.5) and (6.6) are:

$$x_1 = \frac{1}{2}A_0 \cos(\omega_0 t + \psi_0) + \frac{1}{2}A_1 \cos(\omega_1 t + \psi_1), \quad (6.7)$$

$$x_2 = \frac{1}{2}A_0 \cos(\omega_0 t + \psi_0) - \frac{1}{2}A_1 \cos(\omega_1 t + \psi_1), \quad (6.8)$$

where  $\omega_0^2 = k/m$  and  $\omega_1^2 = \omega_0^2 + 2k$ .

The motion of the oscillators is now the superposition of two simple harmonic motions, with the (different) frequencies  $\omega_0$  and  $\omega_1$ . The constants  $A_0$ ,  $A_1$ ,  $\psi_0$  and  $\psi_1$  depend on initial conditions.

The motions are particularly simple for the following cases:

1. Initial conditions

$$\left. \begin{array}{l} \dot{x}_1 = 0, \quad \dot{x}_2 = 0 \\ x_1 = A, \quad x_2 = A \end{array} \right\} \text{at } t = 0.$$

Then

$$x_1 = x_2 = A \cos \omega_0 t. \quad (6.9)$$

Both oscillators vibrate in phase at the same frequency  $\omega_0$ ; this motion is called the “first normal mode of vibration” for our system.

2. Initial conditions

$$\left. \begin{array}{l} \dot{x}_1 = 0, \quad \dot{x}_2 = 0 \\ x_1 = A, \quad x_2 = -A \end{array} \right\} \text{at } t = 0.$$

Then

$$x_1 = -x_2 = A \cos \omega_1 t. \quad (6.10)$$

This is the second normal mode.

## 3. Initial conditions

$$\left. \begin{array}{l} \dot{x}_1 = 0, \quad \dot{x}_2 = 0 \\ x_1 = 0, \quad x_2 = A \end{array} \right\} \text{at } t = 0.$$

Then

$$x_1 = A \sin \frac{1}{2}(\omega_1 - \omega_0)t \cdot \sin \frac{1}{2}(\omega_1 + \omega_0)t, \quad (6.11)$$

$$x_2 = A \cos \frac{1}{2}(\omega_1 - \omega_0)t \cdot \cos \frac{1}{2}(\omega_1 + \omega_0)t. \quad (6.12)$$

❗ **Exercise #2.**

Derive Equations (6.11) and (6.12) from (6.7) and (6.8) with the specified initial conditions.

Equation (6.11) says that the motion of oscillator 1 is a harmonic motion (with frequency  $\frac{1}{2}(\omega_1 + \omega_0)$ ) with an amplitude  $A \sin \frac{1}{2}(\omega_1 - \omega_0)t$  which changes with time; in fact, the amplitude itself is oscillating with a frequency  $\frac{1}{2}(\omega_1 - \omega_0)$ . So at time  $t = 0$  the amplitude is zero; at  $t = \pi/(\omega_1 - \omega_0)$  the amplitude is  $A$ ; at  $t = 2\pi/(\omega_1 - \omega_0)$  the amplitude is zero again and at  $t = 3\pi/(\omega_1 - \omega_0)$  the amplitude is  $-A$ , *etc.*

❗ **Exercise #3.**

Show that the amplitude of oscillator 2 is equal to  $\pm A$  when the amplitude of oscillator 1 is zero, and vice versa.

The two oscillators periodically exchange their kinetic energy; the exchange period is  $T_e = 2\pi/(\omega_1 - \omega_0)$  and the exchange frequency  $\omega_e = (\omega_1 - \omega_0)$ . That is: if at  $t = 0$  oscillator 2 has all the kinetic energy (and oscillator 1 is at rest momentarily), then at time  $t = \pi/(\omega_1 - \omega_0)$  all the kinetic energy is in oscillator 1, (and oscillator 2 is at rest). The “exchange period” is therefore:  $T_e = 2\pi/(\omega_1 - \omega_0)$  and the exchange frequency:  $\omega_e = (\omega_1 - \omega_0)$ .

## 6.2 Single Pendulum

We will now consider the oscillations of one pendulum. Our experimental pendulum is a simple rod, length  $L$ , total mass  $M$ , oscillating around one end. When the pendulum is given a small angular displacement  $\alpha$  from the vertical, a torque  $J$  acts in it.

$$J = -Mgd \sin \alpha, \quad (6.13)$$

where  $g$  is the acceleration of gravity, and  $d$  is the distance from the centre of gravity to the axis of oscillation. Then the equation of motion of the pendulum is

$$I \frac{d^2 \alpha}{dt^2} = -Mgd \sin \alpha, \quad (6.14)$$

where  $I$  is the moment of inertia with respect to the axis of oscillation. For our pendulum,  $I \cong \frac{1}{3}ML^2$  and  $d \cong \frac{1}{2}L$ . If the angle of oscillation  $\alpha$  is small we may approximate  $\sin \alpha \cong \alpha$ , and Equation (6.2) becomes

$$\frac{d^2 \alpha}{dt^2} + \omega_0^2 \alpha = 0, \quad (6.15)$$

where  $\omega_0^2 = Mgd/I$ . We see from Equation (6.2) that the pendulum will execute harmonic oscillations with respect to the vertical with a frequency  $\omega_0$ .

### 6.3 Two Coupled Pendula

Now consider the situation when we couple the motions of two identical pendula by means of a spring with spring constant  $K$ . Let the relaxed spring length be equal to the distance between the axes of oscillation, and let the spring be connected to the pendulum rods at a distance  $h$  from the axes of oscillation (see Figure 6.1). Then the differential equation of motion of pendulum 1 is:

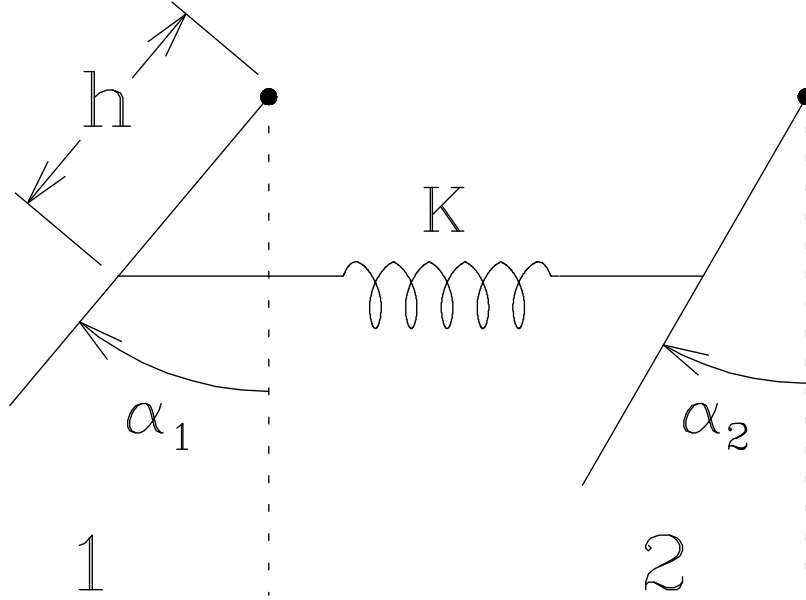


Figure 6.1: Two coupled pendula

$$I \frac{d^2 \alpha_1}{dt^2} = -Mgd \sin \alpha_1 - K(h \sin \alpha_1 - h \sin \alpha_2) \cdot h \cos \alpha_1. \quad (6.16)$$

For small  $\alpha$  this can be written as

$$\frac{d^2 \alpha_1}{dt^2} = -\frac{Mgd}{I} \alpha_1 - \frac{Kh^2}{I} \alpha_1 + \frac{Kh^2}{I} \alpha_2. \quad (6.17)$$

For pendulum 2 the equation of motion is

$$\frac{d^2 \alpha_2}{dt^2} = -\frac{Mgd}{I} \alpha_2 - \frac{Kh^2}{I} \alpha_2 + \frac{Kh^2}{I} \alpha_1. \quad (6.18)$$

ⓘ **Exercise #4.**

Derive Equation (6.17) by considering the torques on the pendulum 1. (Hint: torques are exerted on 1 by its weight and by the extended spring  $K$ .)

Equations (6.17) and (6.18) are identical in form to Equations (6.5) and (6.6) of the general theory, and the theory can now be applied in a straightforward way to our system of two coupled pendula.

ⓘ **Exercise # 5.**

Carefully compare Equations (6.5) and (6.6) with Equations (6.17) and (6.18). Find expressions for  $\omega_0$ ,  $\omega_1$  and  $k$  from the general theory in terms of  $I$ ,  $K$ ,  $M$ ,  $d$  and  $h$ .

**Procedure**

1. Remove the coupling spring from two adjacent pendula. For each pendulum, measure and calculate its natural frequency  $\omega_0$ . If necessary, tune the two pendula by adjusting the tuning screw at the bottom-end of the pendulum rod, so that the uncoupled pendula have identical frequencies.
2. Couple the two pendula with the spring at a distance  $h = 10$  cm. Measure the frequencies  $\omega_0$  and  $\omega_1$  of the two normal modes, and the exchange or beat frequency  $\omega_e$ .
3. Increase  $h$  in steps of 10 cm, up to  $h = 70$  cm and for each  $h$  measure  $\omega_0$ ,  $\omega_1$  and  $\omega_e$ .
4. According to the theory, you should find:
  - $\omega_e = (\omega_1 - \omega_0)$
  - $\omega_1$  proportional to  $h^2$ .

Do your data obey these relations?