Experiment 1

Data Analysis

1.1 Introduction

Some degree of uncertainty exists in any measurement. This uncertainty is called the ERROR. It is to be distinguished from the notion of a mistake and it is not the discrepancy between the student-measured value and that given in a textbook. Rather, it is a quantitative indication of the probability that a further measurement under the same conditions would give a result that falls within a specified range of the reported value. The determination of the error associated with a measurement is part of the general problem of data analysis. In this introductory session we can only hope to touch upon a few of the important concepts.

1.2 Instrumental Error and Significant Figures

An error may be associated with a single measurement, in which case it is a reflection of the limitations on the precision of the measuring device.

Consider a measurement of time, using a stopwatch on which the smallest scale division is 0.2 s. If a single reading of, say, the time taken for a glider to travel over a marked length of an air track is determined as 9.2 s, then it is properly recorded as

$$t = 9.2 \pm 0.1 \text{ s}. \tag{1.1}$$

Equation (1.1) indicated that the observed time lies closer to 9.2 than to 9.0 or 9.4 s, but could have any value between 9.1 and 9.3 s; see Figure 1.1(a).

In this case, the uncertainty in the time is due to the limitation of the instrument; the *instrumental error* δt is then

$$\delta t = \pm 0.1 \text{ s},$$

and the reported value has two significant figures.

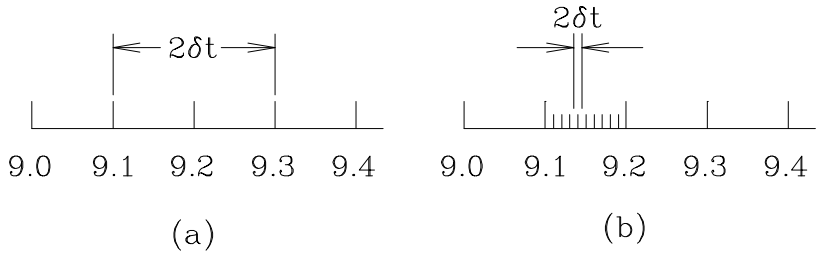

Figure 1.1: Comparison of instrumental error ((a) stopwatch vs. (b) electronic timer)

Assume the stopwatch is replaced with an electronic timer, automatically started and stopped as the glider passes two selected points on the air track. If the smallest time division is now 0.01 s, the result of a measurement might be recorded as

$$t = 9.140 \pm 0.005 \text{ s}. \tag{1.2}$$

The number of significant figures is now 3, reflecting the smaller instrumental error of $\delta t = 0.005$ s. Thus, as shown in Figure 1.1(b), Equation (1.2) indicates that the observed time lies between 9.135 and 9.145 s.

With some instruments, a result is read from a continuously marked dial (this is an "analogue" read-out while the former is a "digital" read-out). In such cases an investigator may be able to decrease the instrumental uncertainty below the limit imposed by the smallest scale division by interpolation.

Table #1		**Table #2**	
Trial #1	**Length (cm)**	**Trial #2**	**Time (seconds)**
1	88.90	1	1.88
2	88.88	2	1.86
3	88.92	3	1.88
4	88.91	4	1.90
5	88.89	5	1.88
6	88.90	6	1.92
7	88.88	7	1.88
8	88.91	8	1.90
9	88.91	9	1.90
10	88.93	10	1.90
11	88.89		
12	88.91		

(!) **Exercise #1.**
In an experiment the value of the gravitational acceleration g is measured by determining the period T of a pendulum of length L. g can be calculated from L and T according to:

$$g = \frac{4\pi^2 L}{T^2}.$$

Many independent measurements of L and T (for the same pendulum) are made; the data collected are shown in Tables 1 and 2.

[?] What instrumental errors would you infer for the measurements of the length and the time?

1.3 Random Errors

In those cases in which only a single measurement is possible, the error reported is necessarily the instrumental error described in Section 1.2.

If the situation permits, however, it is good practice to repeat a measurement several times. If the precision of the instrument is low, it may be found that all measurements are found to be the same within the large instrumental error.

On the other hand, if the instrumental error is reduced by going to a more precise measuring device, it is frequently found that the values so determined differ from one another by more than the instrumental error and these differences generally become more noticeable as the accuracy of the instrument increases. The differences may arise from a number of causes, all of which are characterized by the fact that within the given conditions of the measurement there are random or experimentally uncontrollable variations. One of these, for example, may be due to variation in the judgement of the investigator if he uses interpolation between smallest scale divisions. In other cases, it may arise from slight temperature changes or vibrations, or lack of humidity control, *etc.* Finally, in an important class of measurements, the parameter concerned may have an intrinsic variability; for example, the number of radioactive atoms, in a given sample N_0, that decay during a time interval t, or the number of electrons emitted per unit time from a filament at temperature T, and so on.

In all of these cases the resultant uncertainty in the reported value of the quantity measured can best be represented in statistical terms.

Assume we wish to determine the value of some quantity q and we set out to do so by making a set of n measurements, all made under identical conditions so far as they may be controlled, and all having the same associated instrumental error. We call such a set a "sample." It may be shown (and is intuitively "obvious") that the single parameter that best represents the sample is the arithmetic mean, defined by

$$\langle q \rangle = \frac{1}{n} \sum_{j=1}^{n} q_j. \tag{1.3}$$

The individual q_j's, of course, contain the number of significant figures consistent with the instrumental error.

Stating $\langle q \rangle$ alone, however, is not sufficient to characterize the sample of n measurements. This can be appreciated intuitively by reference to Figure 1.2, in which the results of three sets, each of n measurements, all having the same average value, are represented as frequency histograms. These are constructed by plotting along the ordinate axis the number of times n_i the measured quantity q falls in the range $q_1 \pm \Delta q_i$, against the associated value of q_i plotted along the abscissa axis. The latter is marked off in segments or "bins" of width $2\delta q$. It is clear that if we are to represent the n measurements by the average $\langle q \rangle$, we require to indicate, quantitatively, the degree of "spread" of values about $\langle q \rangle$.

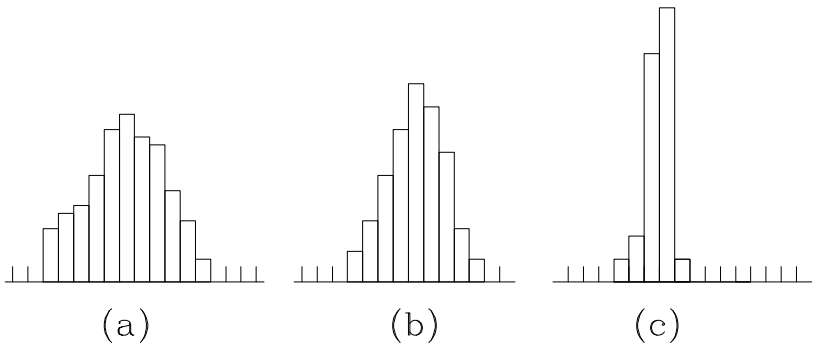

Figure 1.2: Histograms with Gaussian shape

Exercise #2.
From the data in Tables 1 and 2, calculate $\langle L \rangle$ and $\langle T \rangle$. Construct histograms for L and T.

The shape of the histograms shown in Figure 1.2 is called a Gaussian; it arises whenever the variations about the mean are random. We shall restrict ourselves to such cases, which are common, but by no means unique.

Since the individual values of q are distributed about $\langle q \rangle$, we can define the *deviation* of say the j^{th} value q_j from the mean as

$$\Delta q_j = (q_j - \langle q \rangle). \tag{1.4}$$

We might be tempted to use the average of the deviations, *i.e.*,

$$\langle \Delta \rangle = \frac{1}{n} \sum_{j=1}^{n} \Delta q_j,$$

as a measure of the spread of q values in our sample. However, as it will be shown in Exercise 3, this is not a fruitful approach. Instead, we introduce the *DISPERSION* of the sample as a measure of the spread, defined as

$$\langle (\Delta q)^2 \rangle = \frac{1}{n} \sum_{j-1}^{n} (q_j - \langle q \rangle)^2. \tag{1.5}$$

A linear measure of the spread for the sample of n measurements is given by the *STANDARD DEVIATION OF THE SAMPLE*, S, defined by

$$S = \sqrt{\langle (\Delta q)^2 \rangle}. \tag{1.6}$$

Exercise #3.
Show that the average deviation $\langle \Delta q_j \rangle$ is always zero. Calculate the standard deviation of the sample, S from the data in Tables 1 and 2; call the results S_L and S_T.

S is a measure of the width of the histogram; a large S value corresponds to a wide spread of values around the average value. If one further measurement of q was done, there is a 68% probability that it will fall in the range $\langle q \rangle \pm S$, a 95% chance that it will fall in the range $\langle q \rangle \pm 2S$. S is therefore an indication of the precision of a *single* measurement.

In principle, however, we want a measure of the probability that our sample mean q lies within a certain range of the "true value" which we'd obtain if we were to make many, many repeated *samples*, calculate the mean in each case and see how they are distributed. The standard deviation of that distribution is called the *STANDARD DEVIATION OF THE MEAN*, σ_m, and statistical theory shows we can calculate σ_m from S and n by:

$$\sigma_m = \frac{S}{\sqrt{n-1}}. \tag{1.7}$$

Clearly, as we increase the number of measurements in our sample, the mean of that sample will be closer to the true value.

Note this difference: S gives the precision of *a single* measurement, whereas σ_m gives the precision of the *average* of *many* (n) measurements. Obviously, the average value is more precise than the value obtained from one measurement; therefore σ_m is smaller than S.

Exercise #4.
Calculate $\sigma_m(L)$ and $\sigma_m(T)$.

In summary, the error of a single measurement is determined by the precision of the instrument. The error of the result from a sample of n measurements, expressed as the mean $\langle q \rangle$, is expressed in terms of σ_m and quoted as

$$q = \langle q \rangle \pm \sigma_m.$$

If the distribution of q values about the mean is Gaussian, then the probability that $\langle q \rangle$ lies within $\pm\sigma_m$ of the "true" value is 68.3%; the probability that it lies within $\pm 2\sigma_m$ of the "true" value is 95.45%, *etc.*

1.4 The Propagation of Errors

Very frequently a parameter of interest is not measured directly, but is deduced from one or more parameters that are measured, through some functional relationship. For example, the gravitational constant g is approximately given in terms of length L of a pendulum and the period T of a swing by

$$g = \frac{4\pi^2 L}{T^2}.$$

If each of the latter have errors associated with them, we require to know how these errors are propagated to give rise to a corresponding uncertainty in the parameter of interest.

In the following we shall consider a general case, in which the quantity Z is related to two directly measureable parameters, x, y, in the form

$$Z = Z(x, y). \tag{1.8}$$

We assume $\langle x \rangle$ and σ_x have been determined, as in Section 1.3, through n_x measurements, and $\langle y \rangle$ and σ_y through n_y measurements. The reported value of Z is then that obtained from

$$\langle Z \rangle = Z(\langle x \rangle, \langle y \rangle). \tag{1.9}$$

Exercise #5.
Calculate the average value of g.

We need to find σ_Z, the uncertainty in Z arising from σ_x and σ_y. Consider $Z_{jk} = Z(x_j, y_k)$, *i.e.*, the Z value associated with the j^{th} measured value of x and the k^{th} measured value of y. The deviation of Z_{jk} from the mean value is then

$$\Delta Z_{jk} = (Z_{jk} - \langle Z \rangle). \tag{1.10}$$

Now by definition,

$$\Delta x_j = x_j - \langle x \rangle, \quad \Delta y_k = y_k - \langle y \rangle.$$

Thus

$$x_j = \langle x \rangle + \Delta x_j, \quad y_k = \langle y \rangle + \Delta y_k,$$

and

$$Z_{jk} = Z(\langle x \rangle + \Delta x_j, \quad \langle y \rangle + \Delta y_k). \tag{1.11}$$

Z_{jk} can be expanded as a Taylor's series about $Z(\langle x \rangle, \langle y \rangle)$ in the form:

$$\begin{aligned} Z_{jk} &= Z(\langle x \rangle + \Delta x_j, \langle y \rangle + \Delta y_k) \\ &= Z(\langle x \rangle, \langle y \rangle) + \left(\frac{\partial Z}{\partial x}\right) \Delta x_j + \left(\frac{\partial Z}{\partial y}\right) \Delta y_k + \ldots . \end{aligned} \tag{1.12}$$

Here $(\partial Z/\partial x)$ is the partial derivative of Z with respect to x, y being held constant during differentiation. The resulting function is then evaluated at $(\langle x \rangle, \langle y \rangle)$. Similar remarks apply to $(\partial Z/\partial y)$.

Exercise #6.
Evaluate $(\partial g/\partial L)_T$ and $(\partial g/\partial T)_L$ from $g = 4\pi^2 L/T^2$. Calculate the numerical values of these partial derivatives for $L = \langle L \rangle$ and $T = \langle T \rangle$.

Substituting (1.12) into (1.10) gives

$$\Delta Z_{jk} = \left(\frac{\partial Z}{\partial x}\right) \Delta x_j + \left(\frac{\partial Z}{\partial y}\right) \Delta y_k.$$

To obtain the dispersion in Z, i.e. $\langle(\Delta Z)^2\rangle$, we require

$$(\Delta Z_{jk})^2 = \left(\frac{\partial Z}{\partial x}\right)^2 (\Delta x_j)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 (\Delta y_k)^2 + 2\left(\frac{\partial Z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\Delta x_j \Delta y_k. \tag{1.13}$$

We now sum over the x's and the y's to obtain the dispersion:

$$\langle(\Delta Z)^2\rangle = \frac{1}{n_x n_y}\sum_{j=1}^{n_x}\sum_{k=1}^{n_y}(\Delta Z_{jk})^2.$$

This double sum is a bit tricky; the result is:

$$\langle(\Delta Z)^2\rangle = \left(\frac{\partial Z}{\partial x}\right)^2 S_x^2 + \left(\frac{\partial Z}{\partial y}\right)^2 S_y^2. \tag{1.14}$$

The quantities S_x and S_y have been defined previously; recall from equations (1.5) and (1.6) that

$$S_x^2 = \langle(\Delta x)^2\rangle = \frac{1}{n_x}\sum_{j=1}^{n_x}(\Delta x_j)^2,$$

ditto for S_y. Now we define again the standard deviation of the sample as:

$$S_z = \sqrt{\langle \Delta Z^2\rangle} = \sqrt{\left(\frac{\partial Z}{\partial x}\right)^2 S_x^2 + \left(\frac{\partial Z}{\partial y}\right)^2 S_y^2}. \tag{1.15}$$

(!) **Exercise #7.**
From the previously calculated values of S_L, S_T, and the partial derivatives, calculate S_g.

Finally, the precision of the average value of Z, $\langle Z\rangle$, is given by

$$\sigma_z = \frac{S_z}{\sqrt{n_x n_y - 1}}. \tag{1.16}$$

(!) **Exercise #8.**
Calculate σ_g, and compare it to $\langle g\rangle$.

We can now extend this treatment to the case of Z being a function of many variables, $Z = Z(x, y, p, q, \ldots)$, and calculating S_z from $S_x, S_y, S_p, S_q \ldots$. The resulting formulae are rather cumbersome; useful approximations are as follows:

Case A. $Z = ax + by$, a and b are constants.
Equation (1.15) would give:

$$S_z = \sqrt{a^2\, S_x^2 + b^2\, S_y^2}.$$

The right-hand-side is less than

$$[a^2\, S_x^2 + 2ab\, S_x\, S_y + b^2\, S_y^2]^{\frac{1}{2}} = (a\, S_x + b\, S_y).$$

Therefore, a pessimistic estimate of S_z is

$$S_z \simeq aS_x + bS_y.$$

Generalization: if Z is a "sum-type" function of the variables x, y, p, q, like

$$Z = ax + by - kp + \cdots,$$

then

$$S_z \simeq |aS_2| + |bS_y| + |kS_p| + \cdots.$$

Case B. $Z = x^p\, y^q$, p and q are constants.
Then $\left(\frac{\partial Z}{\partial x}\right) = p\, x^{p-1}\, y^q$, $\left(\frac{\partial Z}{\partial y}\right) = x^p\, q\, y^{q-1}$, and equation (1.15) would give:

$$S_z = \sqrt{(p\, x^{p-1}\, y^q)^2\, S_x^2 + (p^x\, q\, y^{q-1})^2\, S_y^2}.$$

The right-hand-side is less than

$$\sqrt{(p\, x^{p-1}\, y^q)^2\, S_x^2 + 2(p\, x^{p-1}\, y^q)(x^p\, q\, y^{q-1})\, S_x\, S_y + (x^p\, q\, y^{q-1})^2\, S_y^2} = (p\, x^{p-1}\, y^q)\, S_x + (x^p\, q\, y^{q-1})\, S_y.$$

Therefore, a pessimistic estimate of S_z is:

$$S_z \simeq (p\, x^{p-1}\, y^q)\, S_x + (x^p\, q\, y^{q-1})\, S_y.$$

This can be written as :

$$\frac{S_z}{Z} = p\frac{S_x}{x} + q\frac{S_y}{y}.$$

Generalization: if Z is a "product-type" function of the variables x, y, r, s, like

$$Z = \frac{x^p\, y^q \ldots}{r^t\, s^m \ldots},$$

then

$$\frac{S_z}{Z} \simeq \left|p\frac{S_x}{x}\right| + \left|q\frac{S_y}{y}\right| + \left|t\frac{S_r}{r}\right| + \cdots.$$

In words: "For a 'sum-type' function the errors are additive; for a 'product-type' function the relative errors are additive."

(!) **Exercise #9.**
Calculate S_g from S_L and S_T using these approximations; compare the result with the exact result obtained in Exercise #7.