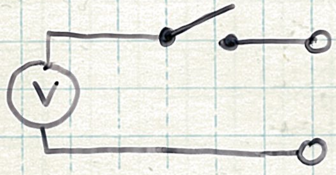
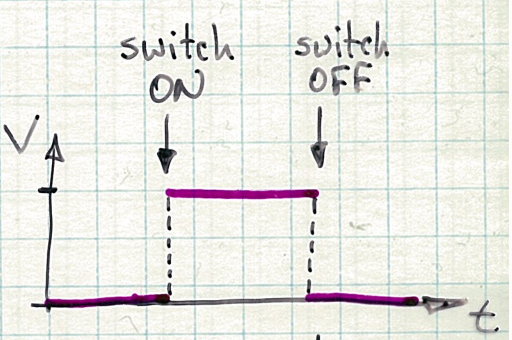
Circuit transients

- Time-dependent voltages:

e.g. a switch

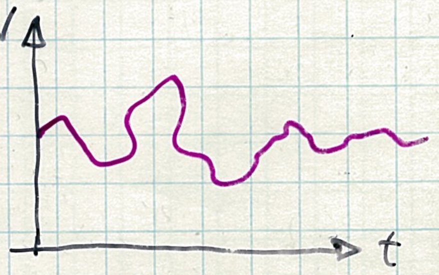




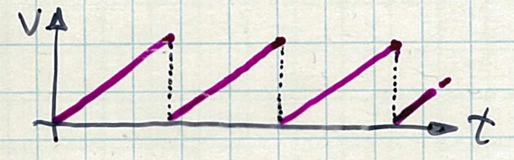
$I = \frac{V}{R}$ ← nothing about the time scale when V is changed

⇒ instant adjustment

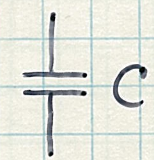
- In general, arbitrary $V(t)$:



  - step functions (see switch)
  - sinusoidal, periodic in time → AC ← alternating
  - sawtooth

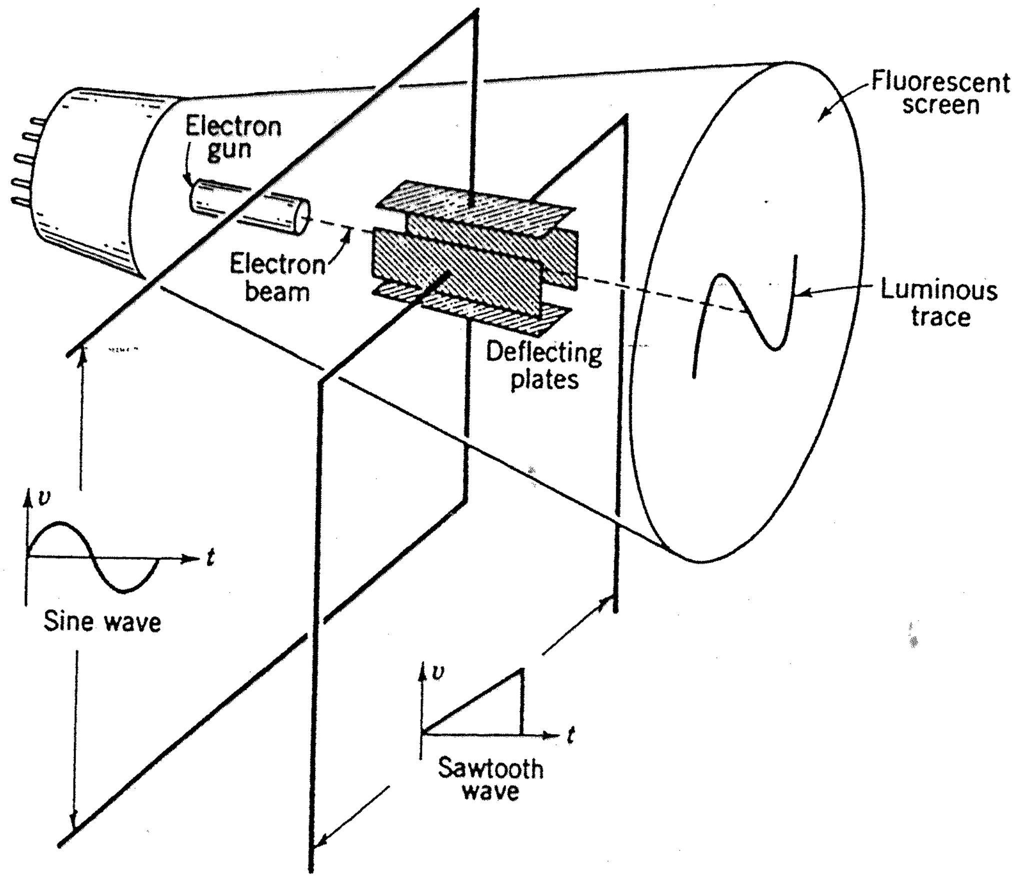


- Other circuit elements that do not respond instantly to changes in V:
  - Capacitors (condensers)

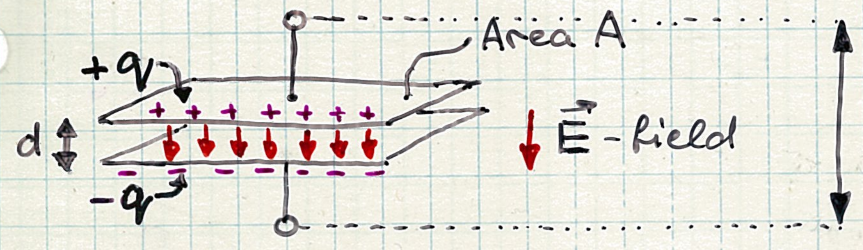


  - inductors (coils, chokes, solenoids)

- Measuring time-dependent voltages
  - meters: V_{min}, V_{max}, V(now!), $V_{average}$
  - oscilloscope (CRT): $V(t)$


Electron gun
Electron beam
Deflecting plates
Fluorescent screen
Luminous trace
v
t
Sine wave
v
t
Sawtooth wave

CAPACITORS



Voltage V : $E = \frac{V}{d}$

- Gauss's Law $\Rightarrow$ plates carry equal and opposite charge

$$q = \varepsilon A E = \varepsilon A \frac{V}{d}$$

permittivity (depends on material)

dependence on A, V, d seems reasonable!

$\varepsilon_0 = 8.85 \times 10^{-12}\ C\, m^{-1} V^{-1}$ in Vacuum

- Vary charge $\Rightarrow$ current $I = \frac{dq}{dt} = \frac{\varepsilon A}{d} \frac{dV}{dt}$

N.B. This is a displacement current, not a continuous flow of charge, like in DC circuits.

- Capacitance $C = \frac{\varepsilon A}{d}$ for a capacitor made of 2 parallel plates

$C \equiv \frac{q}{V}$ ← operational definition, placing q causes V.

$[C] = 1F = \frac{1C}{1V}$ F = Farad

1F is BIG! Typically, 1pF → 1mF

- In general, any two conductors (not just flat plates) separated by an insulator have capacitance.
  - stray capacitance in circuits
  - capacitive proximity sensors

capacitors in DC circuits:

- Steady-state DC :  $\equiv$ open circuit!

  $V = \text{const} \Rightarrow I \propto \frac{dV}{dt} = \emptyset. \Rightarrow$ no energy dissipation

- Transients: add charge to a capacitor

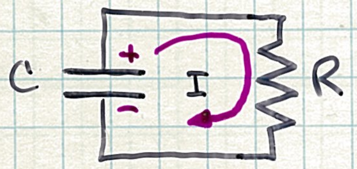
  $+q: \quad \emptyset \rightarrow q$

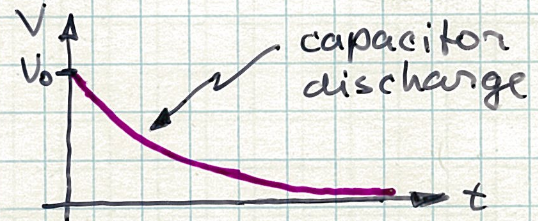
  $V: \quad \emptyset \rightarrow V_0 \quad [= q/C]$

  work $W = \int V I \, dt = \int_0^{V_0} V C \frac{dV}{dt} dt = \frac{1}{2} C V_0^2$

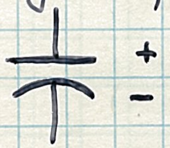
  $\Rightarrow$ energy stored in a capacitor (in E-field)

  $\Rightarrow$ a capacitor connected to a DC circuit behaves like a voltage source for a short time:





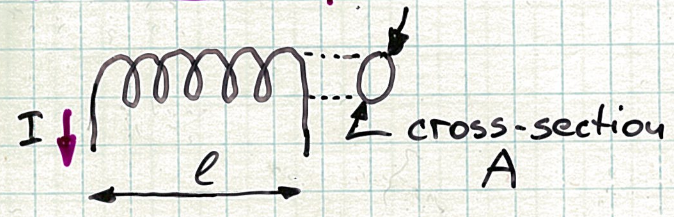
- relative permittivity $\varepsilon/\varepsilon_0$ = dielectric constant
  $\sim 10^5$ for ceramic materials

- electrolytic capacitors : Al foil + conducting paste
  dielectric = Al oxide $\Rightarrow$ high C, but dry out
  polarity important: 

- variable capacitors : change distance or overlap of plates

Inductors

coils, chokes, solenoids



Magnetic flux $\Phi = BA = \frac{\mu N I A}{l}$

μ = permeability ($\simeq \mu_0$ for most materials, except ferromagnetics)

$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H m^{-1}}$ (definition)

- Faraday's law: $V = N \frac{d\Phi}{dt} = \underbrace{\frac{\mu N^2 A}{l}}_{=L\text{, inductance}} \frac{dI}{dt}$

  or: $V = L \frac{dI}{dt}$

$[L]$ = Henries, $1\,\mathrm{H} = \frac{1\,\mathrm{V}}{1\,\mathrm{A}/1\,\mathrm{s}} = 1\,\mathrm{V A^{-1} s}$

- Typical inductors: $1\,\mu\mathrm{H} \rightarrow 1\,\mathrm{H}$

  For example: $\left.\begin{array}{l}\text{3 turns/mm} \\ \text{33.6 cm diameter} \\ \text{1 m long}\end{array}\right\}$ 1 H

- In general, all components have some inductance (esp. wire-wound resistors)

- Symbols:

fixed

variable

- Work $W = \int VI\,dt = \int_0^{I_0} I L \frac{dI}{dt}\,dt = \frac{1}{2} L I_0^2$

  → energy stored in the magnetic field

- Steady-state DC:

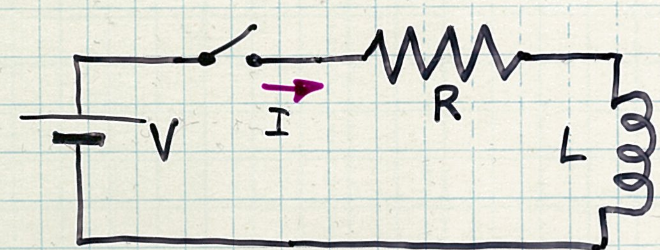
$\equiv$ short circuit, a wire

- $\left.\begin{array}{l} I = C\frac{dV}{dt} \\ V = L\frac{dI}{dt}\end{array}\right\}$ is what makes capacitors and inductors important in circuits.

Inductors in circuits with transients

Lenz's Law: current flow is in a direction which would oppose the flux change causing the e.m.f.

The sign: $V = -L \frac{dI}{dt}$



close the switch:

$V_R = IR$

$V_R = V - L \frac{dI}{dt}$

$\Rightarrow \quad V = IR + L \frac{dI}{dt}$

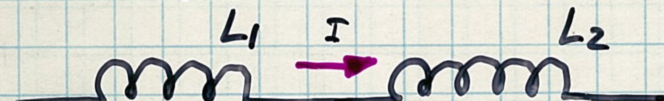
After reaching steady-state, open the switch.

NB: the inductor will oppose the change in the current, so it will try to keep the current going! It will draw on the energy stored in its magnetic field.

Inductors and capacitors in combination

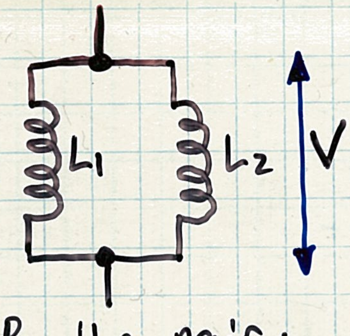
Circuit reduction rules similar to those for resistive networks:

Series L



$$\left. V = L_1 \frac{dI}{dt} + L_2 \frac{dI}{dt} = (L_1 + L_2) \frac{dI}{dt} \right\} \boxed{L = \sum_i L_i}$$

parallel L :

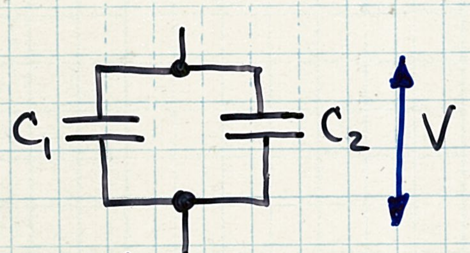


same voltage across both

Current through the pair: $I = \frac{1}{L_1}\int V dt + \frac{1}{L_2}\int V dt$

$$\Rightarrow \quad I = \left(\frac{1}{L_1} + \frac{1}{L_2}\right)\int V dt = \frac{1}{L}\int V dt \quad \Rightarrow \quad \boxed{\frac{1}{L} = \sum_i \frac{1}{L_i}}$$

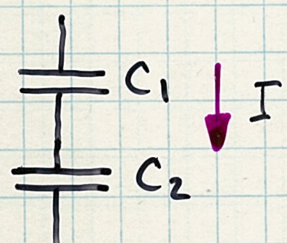
parallel C :



same voltage across both

$$I = C_1\frac{dV}{dt} + C_2\frac{dV}{dt} = (C_1 + C_2)\frac{dV}{dt} \quad \Rightarrow \quad \boxed{C = \sum_i C_i}$$

series C :



same current through both

$$V = \frac{1}{C_1}\int I dt + \frac{1}{C_2}\int I dt = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\int I dt \Rightarrow \boxed{\frac{1}{C} = \sum_i \frac{1}{C_i}}$$

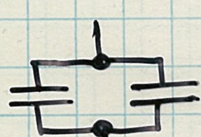
- Intuitive: $\quad L = \frac{\mu N^2 A}{\ell} = \mu A\left(\frac{N}{\ell}\right)^2 \ell \propto \ell$, if $\frac{N}{\ell} = \text{const}$

$\Rightarrow$

$\Rightarrow \quad L = L_1 + L_2$

Also: $\quad C = \frac{\varepsilon A}{d} \propto \text{Area}$

$\Rightarrow$



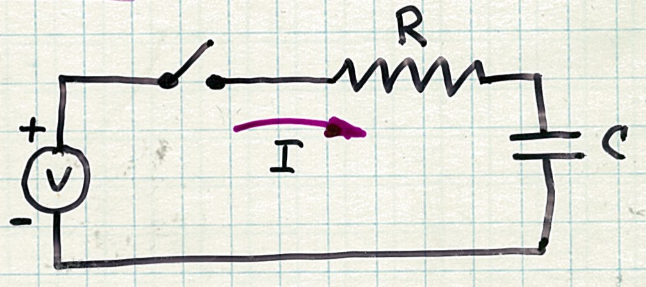
$\Rightarrow \quad C = C_1 + C_2$

- Inductors add according to the rules of resistors, capacitors the reverse.

Transients in RCL circuits

Series RC circuit

switch is closed at $t=\emptyset$



- Kirchhoff's Laws are linear differential equations:

$$V = IR + \frac{1}{C}\int I\,dt, \quad \text{KVL around a single loop}$$

$V = \text{const}$; differentiate to get:

$$\boxed{\frac{dI}{dt} + \frac{1}{RC} I = \emptyset} \qquad (*)$$

- linear eq-n. (I appears only in 1^{st} order)
- involves 1^{st} order derivatives of I
- homogeneous (no terms indep. of I, or RHS $=\emptyset$)

- Solution to such DE is always of the form

$$\underline{I = I_o e^{\alpha t}}$$

Check by substitution: $\quad \frac{dI}{dt} = \alpha I$

$$\alpha I + \frac{1}{RC} I = \emptyset \quad \Rightarrow \quad \underline{\alpha = -\frac{1}{RC}}$$

- Determine I_o from initial conditions: note that the voltage on a capacitor cannot change abruptly (a jump). But initially, across C the voltage $V_c = \emptyset$ (i.e. short circuit)

$$\Rightarrow \quad I(\emptyset) = I_o = \frac{V}{R}$$

Finally, $\boxed{I = \frac{V}{R} e^{-t/RC}}$ solution of (*)

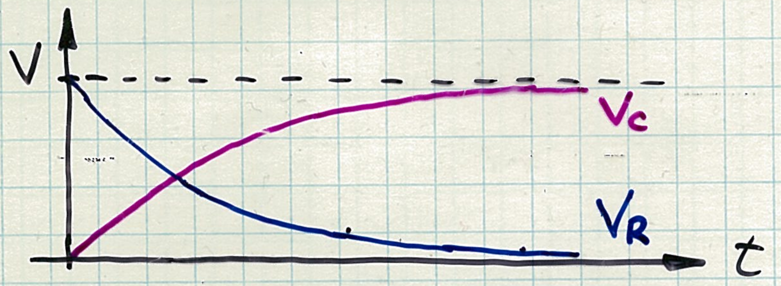
- $\tau \equiv RC$ = a time constant of the circuit, the time to reach new equilibrium

- In terms of voltage,

  – across R : $V_R = IR = V e^{-t/\tau}$

  – across C : $V_C = \frac{1}{C}\int_0^t I\,dt' = V(1 - e^{-t/\tau})$

Check: $V_C + V_R = V$ always. (KVL!)



$t = 0$: $I(0) = I_0 = \frac{V}{R}$ $V_R(0) = V$ $V_C(0) = 0$

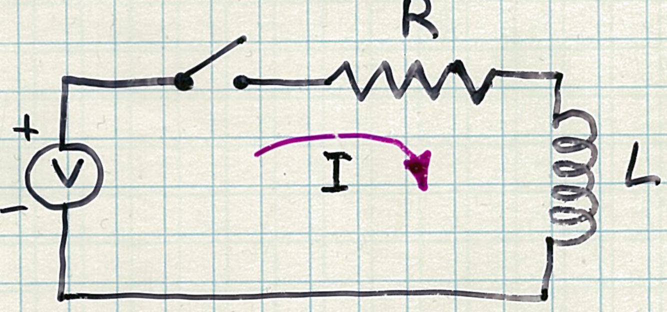
$t = \infty$: $I(\infty) = 0$ $V_R(\infty) = 0$ $V_C(\infty) = V$

"Short time", $t \ll RC$: C behaves like S.C.

"Long time", $t \gg RC$: C behaves like O.C.

Series RL circuit

switch is closed at $t = 0$



For $t \geq 0$, KVL:

$$V = IR + L\frac{dI}{dt}$$

- Rewrite as: $\boxed{\dfrac{dI}{dt} + \dfrac{R}{L} I = \dfrac{V}{L}}$ (**)

  - linear
  - first-order
  - inhomogeneous (RHS $\neq \emptyset$)

  } DE

$\Downarrow$

solution always in two parts: $I = I_h + I_p$

I_h: homogeneous sol-n $\leftrightarrow$ transient

I_p: particular sol-n $\leftrightarrow$ steady-state

- To obtain I_h, set RHS $= \emptyset$: $\dfrac{dI}{dt} + \dfrac{R}{L} I = \emptyset$

$\Rightarrow$ $I_h = I_0 \, e^{-\frac{R}{L}t}$ (as before, with $\tau = \dfrac{L}{R}$)

- Steady-state solution is the constant part of I which accounts for the RHS. Steady-state condition is $t \to \infty$:

$t \gg \dfrac{L}{R} \Rightarrow \dfrac{dI}{dt} = \emptyset$ (steady-state) $\Rightarrow$ $I_p = \dfrac{V}{R}$

i.e. in the long-time limit $L \to$ short circuit.

- Calculate I_0 from the initial conditions: current in an inductor cannot change abruptly, so right after the switch is closed $I(t=\emptyset^+) = \emptyset$.

$I(\emptyset^+) = I_0 \, e^{\emptyset} + \dfrac{V}{R} = \emptyset \Rightarrow I_0 = -\dfrac{V}{R}$

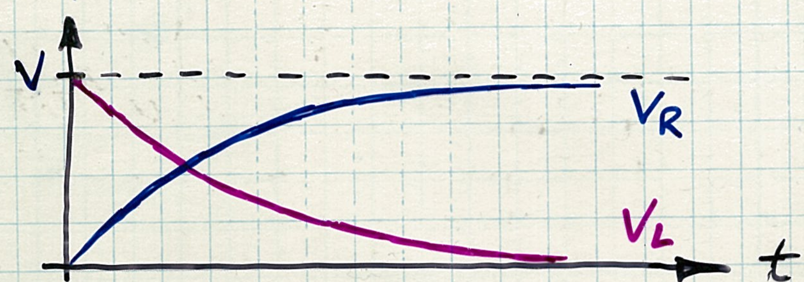
- Finally, $\boxed{I = \dfrac{V}{R}\left(1 - e^{-Rt/L}\right)}$ solution of (**)

- In terms of voltage:

-across R: $V_R = IR = V\left(1 - e^{-Rt/L}\right)$

-across L: $V_L = L\frac{dI}{dt} = Ve^{-Rt/L}$

As always, KVL is satisfied: $V_R + V_L = V, \forall t$



$t=0$: $I(0)=0$ $V_R(0)=0$ $V_L(0)=V$

$t=\infty$: $I(\infty)=I_p=\frac{V}{R}$ $V_R(\infty)=V$ $V_L(\infty)=0$

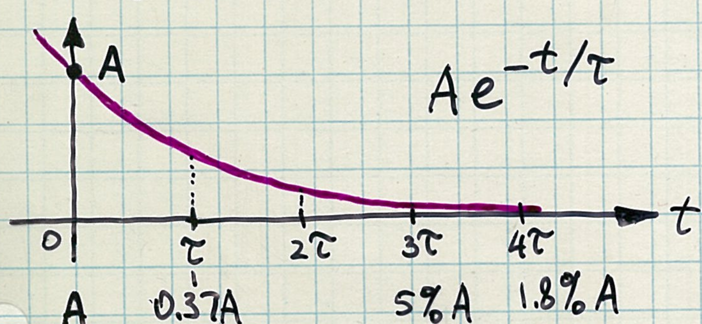
"Short time", $t \ll \frac{L}{R}$: L behaves like O.C.

"Long time", $t \gg \frac{L}{R}$: L behaves like S.C.

NB: C and L behaviours are complementary

Aside

Time constants and exponential decay



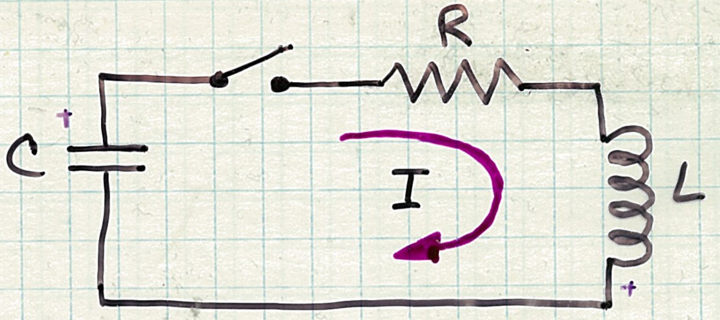
$A \to \frac{1}{2}A$ in 0.693τ

$A \to \frac{1}{e}A$ in τ

Can find τ from 2 points:

$$\left.\begin{array}{l} f_1 = A e^{-t_1/\tau} \\ f_2 = A e^{-t_2/\tau} \end{array}\right\} \quad \tau = \frac{t_2 - t_1}{\ln f_1 - \ln f_2}$$

RCL circuits



C charged to voltage V_0.
Close switch at $t=0$.

- KVL @ $t \geq 0$: $\quad \frac{1}{C}\int I\,dt + IR + L\frac{dI}{dt} = 0$

differentiate: $\quad \boxed{\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = 0}$

$\left.\begin{array}{l}\text{– linear} \\ \text{– 2}^{\text{nd}}\text{ order} \\ \text{– homogeneous}\end{array}\right\}$ DE

Full solution requires **two** initial conditions; here $I(t=0)$ and $\frac{dI}{dt}(t=0)$

- Try the solution $I = I_0 e^{\alpha t}$.

$$\alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} = 0$$

$$\Rightarrow \quad \underline{\alpha_{1,2} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}}$$

and the most general solution is:

$$\underline{I = I_1 e^{\alpha_1 t} + I_2 e^{\alpha_2 t}}$$

with I_1, I_2 determined from the [two] IC's:

– Inductor: $\quad I(t<0) = 0 \Rightarrow I(0^+) = 0$

– $I(0) = 0 \Rightarrow V_R(0) = IR = 0 \Rightarrow V_L = -V_C = V_0$

IC: $\quad I(0) = 0 = I_1 e^0 + I_2 e^0 = I_1 + I_2 \Rightarrow \underline{I_1 = -I_2}$

$$\frac{dI}{dt}(0) = \frac{V_L}{L} = I_1\alpha_1 + I_2\alpha_2 = I_1(\alpha_1 - \alpha_2)$$

$$\Rightarrow \quad I_1 = -I_2 = \frac{V_0}{(\alpha_1 - \alpha_2)L}$$

Finally,

$$\boxed{I = \frac{V_0}{(\alpha_1-\alpha_2)L}\left(e^{\alpha_1 t} - e^{\alpha_2 t}\right)}$$

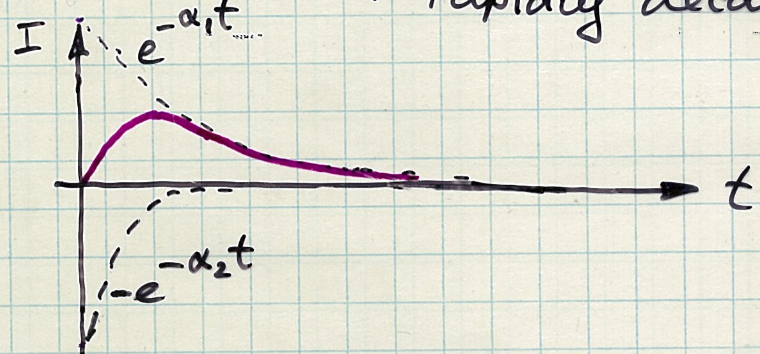
with

$$\alpha_{1,2} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

- **sign** of $\left(\frac{R^2}{4L^2} - \frac{1}{LC}\right)$ determines the nature of the solution.

① $R^2 > 4L/C$ $\Rightarrow$ α_1, α_2 are real and −ve
$|\alpha_2| > |\alpha_1|$

$\Rightarrow$ solution = slowly decaying +ve term } equal
+ rapidly decaying −ve term } magnitu



OVERDAMPING

② $R^2 = 4L/C$ $\Rightarrow$ $\alpha_1 = \alpha_2 = -\frac{R}{2L}$

$$I = \frac{V_0}{L}\,\frac{e^{\alpha_1 t} - e^{\alpha_2 t}}{\alpha_1 - \alpha_2} = \text{undefined } \left(\frac{\varnothing}{\varnothing}\right)$$

Let $\varepsilon = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$ and look at the limit $\varepsilon \to \varnothing$

$$\alpha_1 = -\frac{R}{2L} + \varepsilon, \quad \alpha_2 = -\frac{R}{2L} - \varepsilon \Rightarrow \alpha_1 - \alpha_2 = 2\varepsilon$$

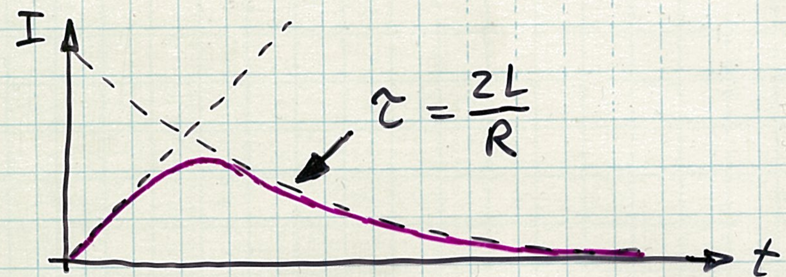
$$\Rightarrow I = \frac{V_0}{L}\,\frac{1}{2\varepsilon}\,e^{-Rt/2L}\left(e^{\varepsilon t} - e^{-\varepsilon t}\right)$$

For small ε ($\varepsilon \to \emptyset$), can use Taylor series' expansion:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \simeq 1 + x$$

$$\Rightarrow \quad e^{\varepsilon t} - e^{-\varepsilon t} \simeq (1+\varepsilon t) - (1 - \varepsilon t) = 2\varepsilon t$$

$$\Rightarrow \quad I = \frac{V_0 t}{L} e^{-Rt/2L}$$



CRITICAL DAMPING

Similar in appearance to overdamping; reaches the axis faster.

③ $R^2 < 4L/C$ $\Rightarrow$ $\alpha_{1,2} = -\frac{R}{2L} \pm \frac{j}{\sqrt{LC}}\sqrt{1 - \frac{R^2 C}{4L}}$

$j = \sqrt{-1}$ | also known as i; use j here to avoid confusion with current

Define

$$\omega \equiv \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{R^2C}{4L}} \simeq \frac{1}{\sqrt{LC}} \quad \text{if } R^2 \ll \frac{4L}{C}$$

usually adequate

$$\Rightarrow \quad I = \frac{V_0}{2j\omega L} e^{-Rt/2L}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

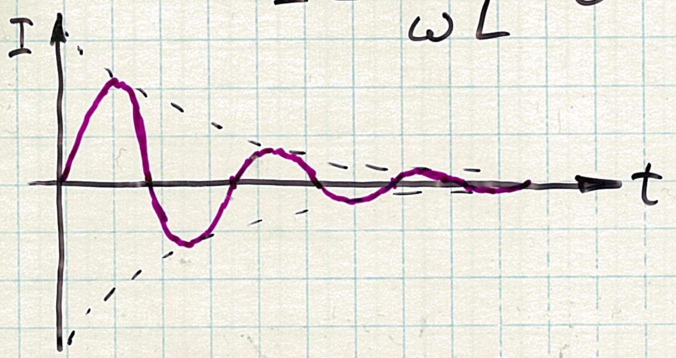
Complex numbers: $e^{j\theta} = \cos\theta + j\sin\theta$

$e^{-j\theta} = \cos\theta - j\sin\theta$

$$\Rightarrow \quad \frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta$$

Hence

$$I = \frac{V_0}{\omega L} e^{-Rt/2L} \sin \omega t$$



UNDERDAMPING

Decays slower than in the critical case.

Oscillations : RCL is the simplest resonance circuit. Frequency of oscillations :

$$[\omega] = \text{radians } s^{-1}$$

$$\omega = 2\pi f = \frac{2\pi}{T} \qquad [f] = s^{-1} = Hz \leftarrow \text{frequency}$$

$$[T] = s \qquad \leftarrow \text{period}$$

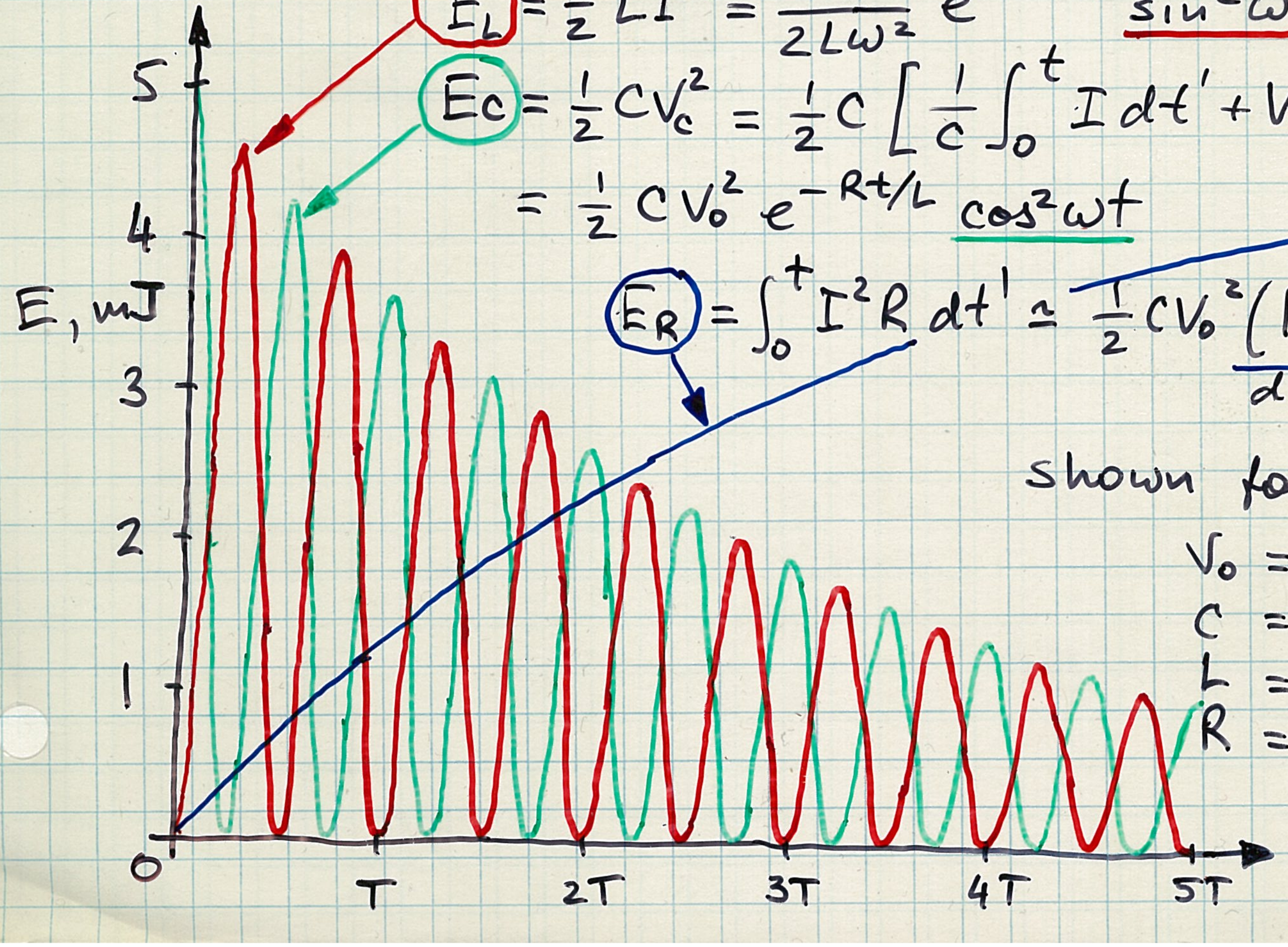
Energy in different parts of the circuit:

$$E_L = \frac{1}{2} L I^2 = \frac{V_0^2}{2L\omega^2} e^{-Rt/L} \sin^2 \omega t$$

$$E_C = \frac{1}{2} C V_c^2 = \frac{1}{2} C \left[\frac{1}{C} \int_0^t I \, dt' + V_0 \right]^2 =$$

$$= \frac{1}{2} C V_0^2 e^{-Rt/L} \cos^2 \omega t$$

$$E_R = \int_0^t I^2 R \, dt' \simeq \underbrace{\frac{1}{2} C V_0^2 \left(1 - e^{-Rt/L}\right)}_{\text{dissipated}}$$



shown for:

$V_0 = 100\,V$
$C = 1\,\mu F$
$L = 0.1\,H$
$R = 100\,\Omega$