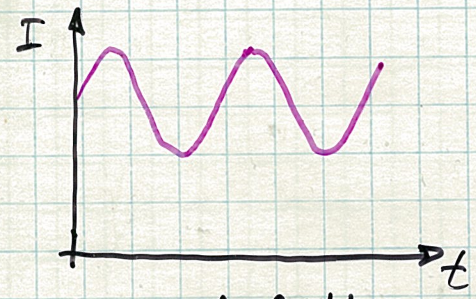
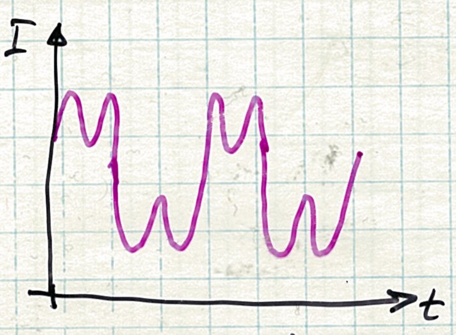
Sinusoidal analysis

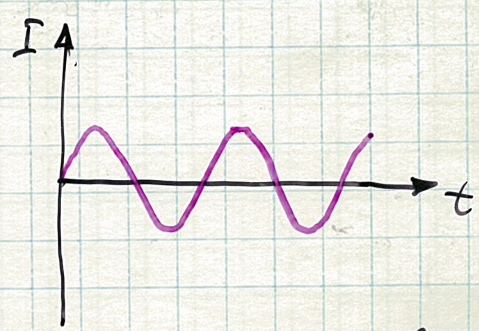


- sinusoidal time-dep.
- I always unidirectional

⇒ DC current + AC ripple

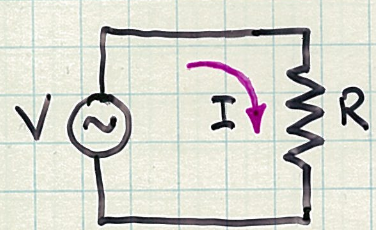


- non-sinusoidal, but periodic time dependence

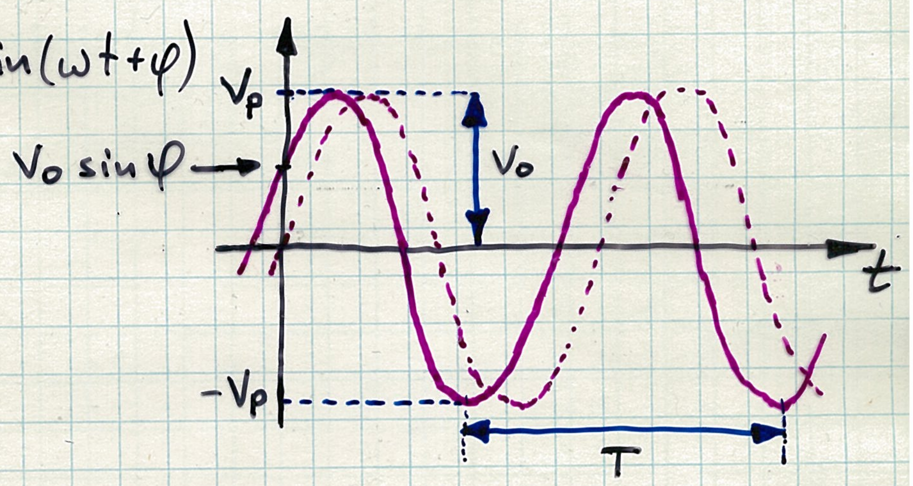


- current alternating direction
- sinusoidal

⇒ AC + ϕ DC offset



$V = V_0 \sin(\omega t + \varphi)$



Terminology

T = period

V_0 = amplitude

V_{pp} = peak-to-peak voltage, $V_p - (-V_p) = 2V_0$

ω = angular frequency (radians/s), $\omega = 2\pi f = \frac{2\pi}{T}$

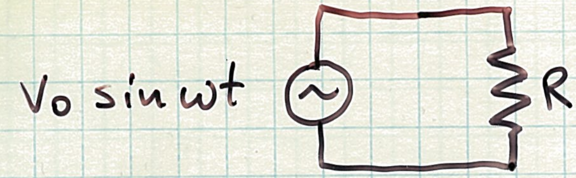
f = frequency, $Hz = s^{-1} = \frac{1}{T}$

φ = phase : dotted line ↔ phase $\varphi = \phi$
solid line ↔ phase $\varphi > \phi$, "ahead"

$\left.\begin{array}{l}\text{solid line leads the dotted line by } \varphi \\ \text{dotted line lags the solid line by } \varphi\end{array}\right\}$ equivalent

Note.
could have used $\cos(\omega t + \varphi) = \sin(\omega t + \varphi')$
with $\varphi = \varphi' + \frac{\pi}{2}$

⇒ a difference of a phase factor.



$$I = \frac{V_0}{R} \sin \omega t$$

- Average power over one period:

$$\bar{P} = \frac{1}{T}\int_0^T I^2 R\, dt = \frac{1}{T}\int_0^T \frac{V_0^2}{R} \sin^2 \omega t\, dt = \frac{V_0^2}{2R}$$

$\Rightarrow$ another way of characterizing the voltage:

$$V_{rms} = \left[\frac{1}{T}\int_0^T V^2(t)\, dt \right]^{1/2} = \frac{V_0}{\sqrt{2}} \simeq 0.707\, V_0$$

Useful because $\bar{P} = \dfrac{V_{rms}^2}{R}$

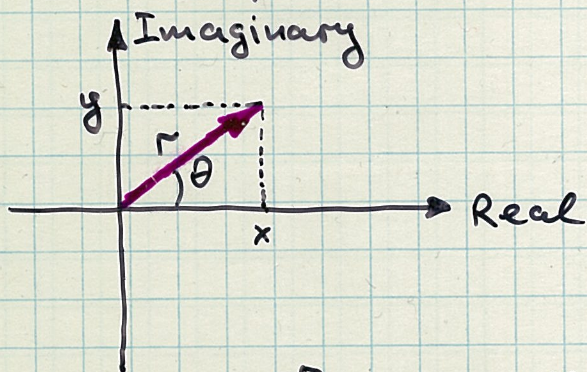
Ex. "110 V AC" $\longrightarrow$ $V(t) = (110 \times \sqrt{2}) \cos \omega t$

Aside

- representation by complex numbers

Euler's formula

$$\boxed{e^{j\theta} = \cos\theta + j \sin\theta}$$



$$j \equiv \sqrt{-1}$$

$$\begin{cases} x = r\cos\theta \\ y = r \sin\theta \end{cases}$$

$$\Rightarrow \quad z = x + jy = r(\cos\theta + j \sin\theta) = r e^{j\theta}$$

$$\sin\theta = \mathrm{Im}(z) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

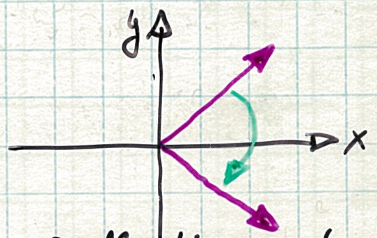
$$\cos\theta = \mathrm{Re}(z) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Operations on complex numbers

conjugation

$$z^* = x - jy$$

$$z^* = r e^{-j\theta}$$



reflection about x

addition

$$z_1 = x_1 + jy_1 \qquad z_2 = x_2 + jy_2$$

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

vector addition

multiplication

$$z_1 z_2 = (x_1 + jy_1)(x_2 + jy_2) =$$

$$= x_1 x_2 + jy_1 x_2 + x_1 jy_2 + j^2 y_1 y_2$$

$$= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1)$$

$$z_1 z_2 = \left(r_1 e^{j\theta_1}\right)\left(r_2 e^{j\theta_2}\right) = r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

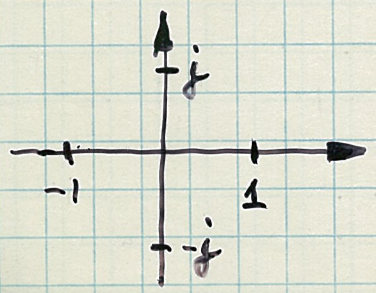
division

$$\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{(x_2 + jy_2)(x_2 - jy_2)}$$

$$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + j\,\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$$

useful relations



$$|z|^2 = z z^* = x^2 + y^2 = r^2$$

$$j^2 = -1 \Rightarrow \frac{1}{j} = \frac{j}{j^2} = \frac{j}{-1} = -j$$

$$j = 0 + j1 = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} = e^{j\pi/2}$$

$$-j = \frac{1}{j} = e^{-j\pi/2}$$

Phasor notation : equivalent to exponential

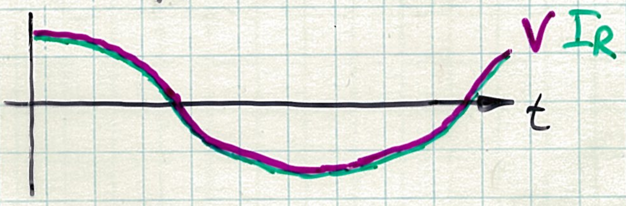
write $z = re^{j\theta}$ as $z = r\angle\theta$

For example, $z_1 z_2 = r_1 r_2 e^{j(\theta_1+\theta_2)} = (r_1 r_2)\angle(\theta_1+\theta_2)$

Resistance

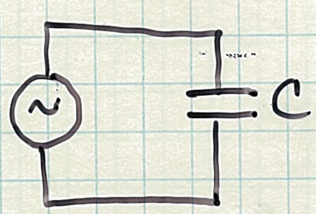
$$V(t) = V_0 \cos\omega t = \mathrm{Re}\left[V_0 e^{j\omega t}\right]$$

$$I_R(t) = \frac{V_0}{R}\cos\omega t$$



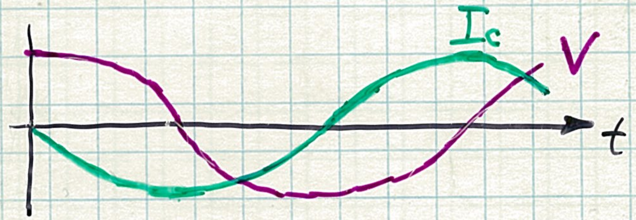
in-phase

Capacitance



$$I_C = C\frac{dV}{dt} = -\omega C V_0 \sin\omega t$$

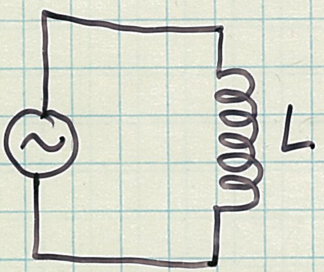
$$= \mathrm{Re}\left[j\omega C V_0 e^{j\omega t}\right]$$



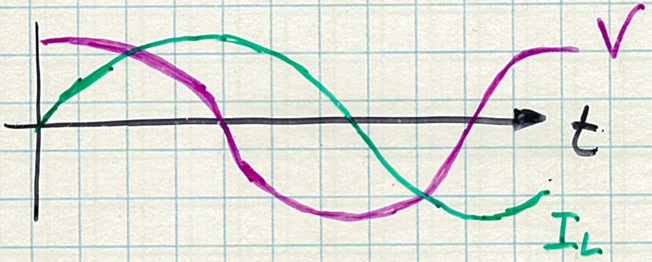
$\Delta\varphi = \frac{\pi}{2}$

I leads V

Inductance



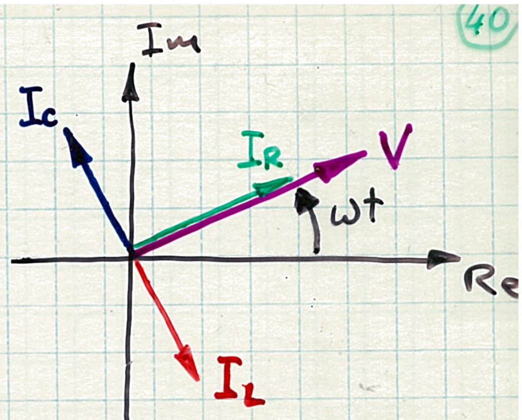
$$I_L = \frac{1}{L}\int V\,dt = \frac{V_0}{\omega L}\sin\omega t$$



$\Delta\varphi = -\frac{\pi}{2}$

I lags V

Summary



phasor diagram

$V = V_0 e^{j\omega t}$

$I_R = \frac{V_0}{R} e^{j\omega t}$

$I_C = j\omega C V_0 e^{j\omega t} = \omega C V_0 e^{j(\omega t + \pi/2)}$

$I_L = \frac{V_0}{j\omega L} e^{j\omega t} = \frac{V_0}{\omega L} e^{j(\omega t - \pi/2)}$

I_C leads V by 90°
I_L lags V by 90°
Only I_R is in-phase with V

This is the so-called frequency-domain representation

In frequency domain, the ratio of V to I is time-independent! ⇒ generalize to the complex form of Ohm's Law:

$$\boxed{V = I Z}$$ AC Ohm's Law

$Z \equiv$ **impedance** $= \frac{V}{I}$ (generalized)

$\boxed{Z_R = R}$ $\boxed{Z_C = \frac{1}{j\omega C}}$ $\boxed{Z_L = j\omega L}$

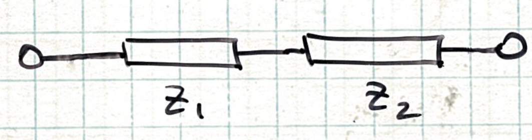
! ⇒ complex algebraic equations replace differential equations needed for time-domain solutions

! ⇒ solutions are complex, with amplitude and phase

! ⇒ impedance is a purely sinusoidal concept, not good for transients

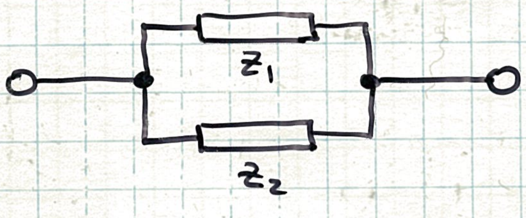
- using impedance, can treat all components as "resistors":

series



$$z = z_1 + z_2$$

parallel



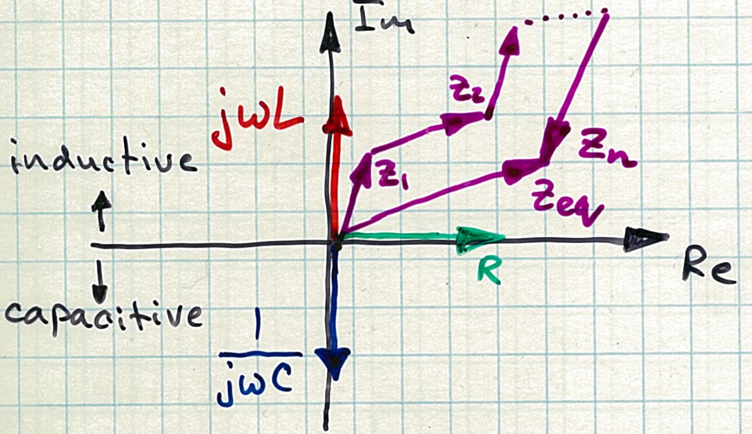
$$\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2}$$

- magnitude of impedance is called reactance

$$X_R = R \qquad X_C = \frac{1}{\omega C} \qquad X_L = \omega L$$

reciprocal of impedance is called admittance
→ angle to x-axis is equal & opposite to that of impedance.

- arbitrary circuit element ⟷ vector (not a phasor! $e^{j\omega t}$ factors out of $\frac{V}{I}$)



$$z_C = \frac{1}{j\omega C} \propto \frac{1}{j} = -j$$

$$z_L = j\omega L \propto j$$

$$z_{eq} = z_1 + z_2 + \ldots + z_n$$
(series connection)

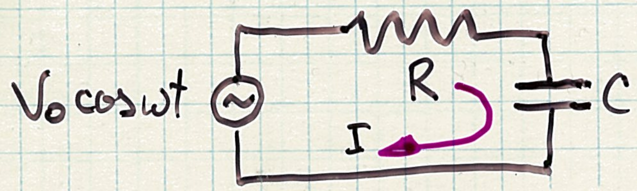
- average power $\boxed{\bar{P} = \frac{1}{T}\int_0^T VI\,dt}$

– $V_C \sim \cos\omega t \Rightarrow I_C = C\frac{dV}{dt} \sim -\sin\omega t \Rightarrow \bar{P} \sim \int_0^T \sin\omega t \cos\omega t = \phi$

– $V_L \sim \cos\omega t \Rightarrow I_L = \frac{1}{L}\int V dt \sim \sin\omega t \Rightarrow \bar{P} = \phi$

→ no energy dissipation – E stored for release $\frac{1}{4}$ cycle later

Time-domain solution



KVL: $V_0 \cos\omega t = IR + \frac{1}{C}\int I\,dt$

$$\Rightarrow \boxed{\frac{dI}{dt} + \frac{1}{RC} I = -\frac{\omega V_0}{R}\sin\omega t}$$ D.E.

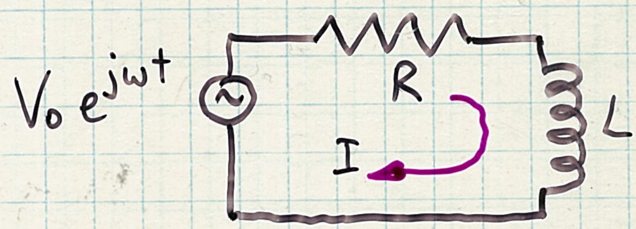
1st order, non-homogeneous, the RHS is time-dependent

Highlights:

1. Guess a general sol-n
$I = I_1 \sin\omega t + I_2 \cos\omega t$
2. Substitute and solve for sin and cos terms separately:
$$I_1 = -\frac{\omega C V_0}{\omega^2 R^2 C^2 + 1}$$
$$I_2 = \frac{\omega^2 R C^2 V_0}{\omega^2 R^2 C^2 + 1}$$
3. Combine as
$$\boxed{I = \frac{\omega C V_0}{\omega^2 R^2 C^2 + 1}(\omega RC \cos\omega t - \sin\omega t)}$$
or
$$I = \frac{V_0}{\sqrt{R^2 + 1/\omega^2 C^2}} \cos(\omega t + \varphi)$$
with $\varphi = \arctan\left(\frac{1}{\omega RC}\right)$
4. $V_R = IR = \ldots$
$V_C = \frac{1}{C}\int I\,dt = \ldots$

c.f.

Frequency-domain solution



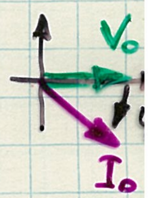
$$Z_{L+R} = R + j\omega L$$

$$I = \underbrace{\frac{V_0}{R + j\omega L}}_{I_0} e^{j\omega t}$$ Algebraic eq-n

$$I_0 = \frac{V_0 \, (R - j\omega L)}{(R + j\omega L)(R - j\omega L)} = \frac{V_0 (R - j\omega L)}{R^2 + \omega^2 L^2}$$

To find phase:
$$x + jy = \sqrt{x^2 + y^2}\, e^{j\varphi}, \quad \varphi = \arctan\frac{y}{x}$$

$$\Rightarrow I_0 = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}} e^{j\varphi}$$



with $\varphi = \arctan\frac{-\omega L}{R}$.

Finally:
$$\boxed{I = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}} e^{j(\omega t + \varphi)}}$$

Transform back to time domain: $e^{j(\omega t + \varphi)} \rightarrow \cos(\omega t + \varphi)$

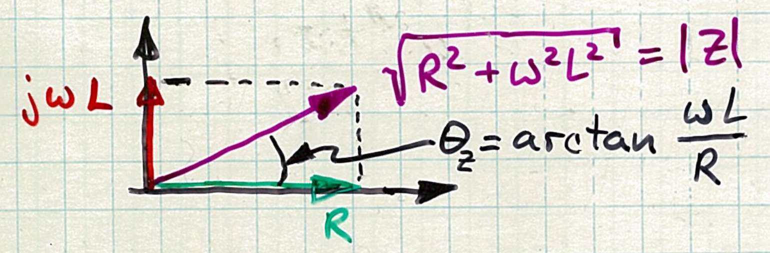
$$\Rightarrow I = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \varphi)$$

with $\varphi = \arctan\frac{-\omega L}{R}$

- **Note 1.** in RL circuit, the current lags voltage by amount intermediate between that for R ($0°$) and pure L ($-90°$)

  Similarly for RC: between R($0°$) and pure C ($+90°$)

- **Note 2** operations on complex numbers:



$$\boxed{I_0 \angle \theta_I = \frac{V_0 \angle \theta_V}{Z_0 \angle \theta_Z} = \frac{V_0}{Z_0} \angle \theta_V - \theta_Z}$$

Here: $\theta_V = \omega t$, $\theta_Z = -\varphi$

$\Rightarrow \theta_I = \omega t + \varphi$

- **Note 3** Electrical engineers' struggle for "$\cos\varphi$"

$$\text{Power factor} = \frac{\text{Power dissipated by } Z}{\text{Apparent power} = V_{rms} I_{rms}}$$

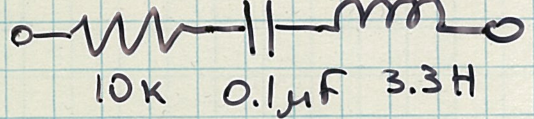
$$= \frac{Re(Z)}{|Z|} = \underline{\cos\varphi}$$

$$|Z| = \left[(Re Z)^2 + (Im Z)^2\right]^{1/2}$$

! no power dissipation in pure C or L

For example, add C to inductive loads (motors) to reduce $Im Z$.

- **Ex.**



$Z_R = R = 10 \times 10^3 = 10^4$

$Z_C = -j \frac{1}{\omega C} = -j \frac{1}{2\pi \times 440 \times 0.1 \cdot 10^{-6}}$

$= -j\,3617$

$Z_L = j\omega L = j\,2\pi \times 440 \times 3.3 = j91$

$Z = Z_R + Z_C + Z_L = \ldots$

$= 10{,}000 + j\,5506\ \Omega$