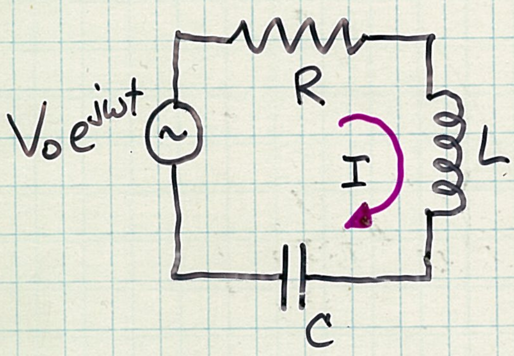
RESONANCE

Series RCL circuit



$$Z = R + j\omega L + \frac{1}{j\omega C}$$

$$I = \frac{V_0 e^{j\omega t}}{Z} = I_0 e^{j\omega t}$$

$$I_0 = \frac{V_0}{Z} = \frac{V_0}{R + j(\omega L - 1/\omega C)}$$

$$\Rightarrow I_0 = \frac{[R - j(\omega L - 1/\omega C)]V_0}{R^2 + (\omega L - 1/\omega C)^2} \overset{*}{=} \frac{V_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} e^{j\varphi}$$

where $\varphi = \arctan\left(\frac{1/\omega C - \omega L}{R}\right)$

* EFTS

$\Rightarrow$ Re-write in time domain:

$$I(t) = \frac{V_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \cos(\omega t + \varphi)$$

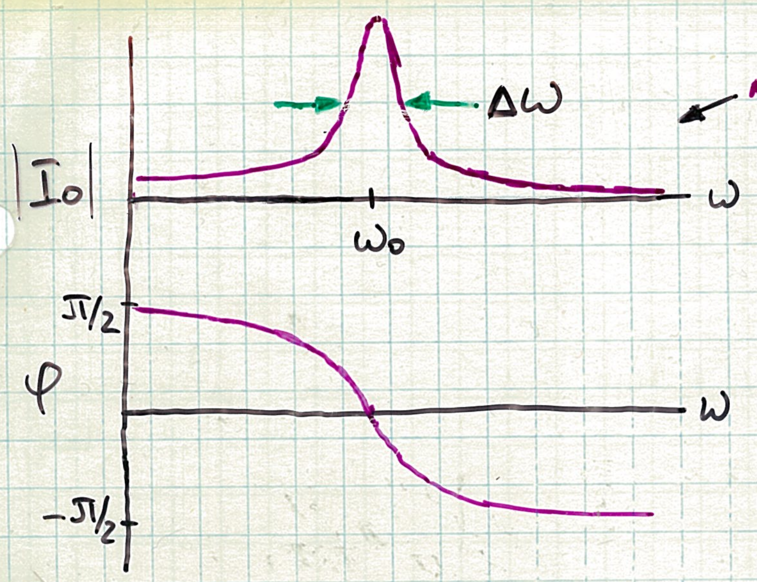
Magnitude of phasor current $|I_0| = \max$, if

$$\omega = \omega_0 = \frac{1}{\sqrt{LC}} \qquad \omega_0 \equiv \text{resonant frequency}$$

At $\omega = \omega_0$: $\varphi = \phi$, $I_0 = \frac{V_0}{R}$ $\Rightarrow$ purely resistive circuit

$\omega < \omega_0$: circuit looks capacitive

$\omega > \omega_0$: circuit looks inductive



NB: not symmetric

At resonance:

$$\omega = \omega_0 = (LC)^{-1/2}$$

$$\varphi = \phi$$

$$Z = R$$

$$I_0 = \frac{V_0}{R}$$

$\Delta\omega$ = difference between half-power points, i.e. where $I = \frac{I_{max}}{\sqrt{2}} \Leftrightarrow \boxed{\omega L - \frac{1}{\omega C} = \pm R}$ (*)

Rewrite (*) and solve:

$$\omega^2 \mp \frac{R}{L}\omega - \frac{1}{LC} = \phi \Rightarrow \begin{cases} \omega_h = +\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \\ \omega_e = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \end{cases}$$

$$\Rightarrow \Delta\omega = \omega_h - \omega_e = \frac{R}{L}$$

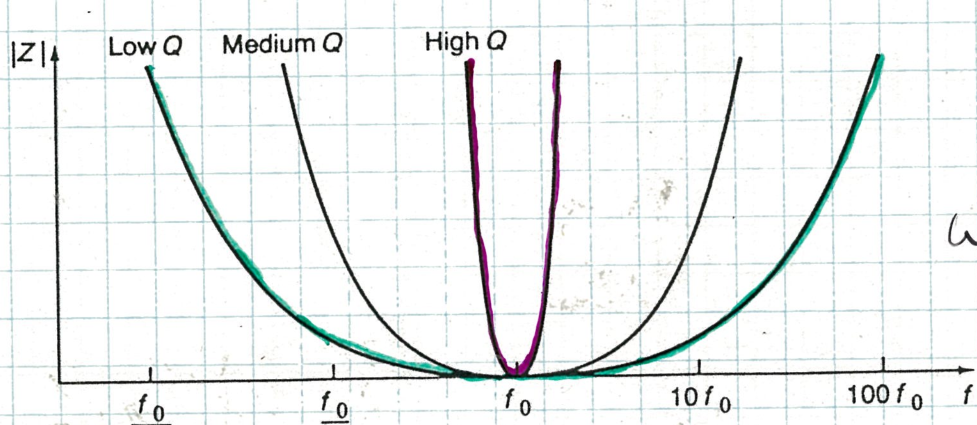
Quality factor of a circuit

$$Q \equiv \frac{\omega_0}{\Delta\omega} = \frac{\omega_0 L}{R}$$

• Note: $Q = \frac{X_L}{R} = \frac{X_C}{R}$ since on resonance:

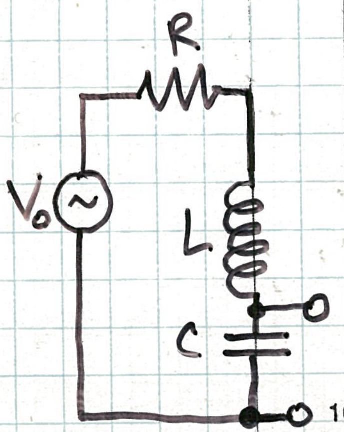
$$X_L \equiv \omega L\Big|_{\omega=\omega_0} = \omega_0 L \quad \text{and} \quad X_C \equiv \frac{1}{\omega C}\Big|_{\omega=\omega_0} = \frac{\omega_0}{\omega_0^2 C} = \frac{\omega_0}{\frac{1}{LC} C} = \omega_0 L$$

• Note: C & L determine ω_0 (frequency)
R & L determine $\Delta\omega$ (width, i.e. the time constant)

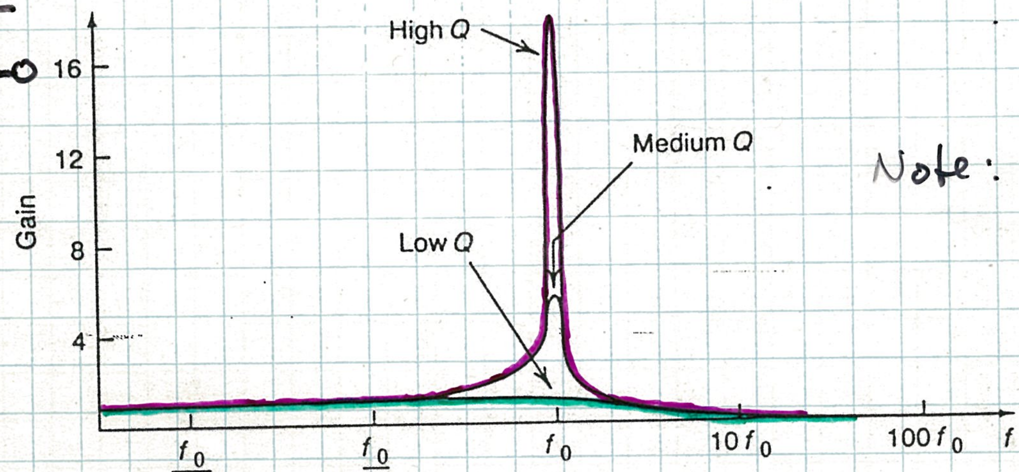


(a)

$$\omega_0 = 2\pi f_0 = \frac{1}{\sqrt{LC}}$$



$$\text{Gain} = \frac{V_c}{V_0}$$



(b)

Note: not symmetric

$\tau_1 > \tau_2$

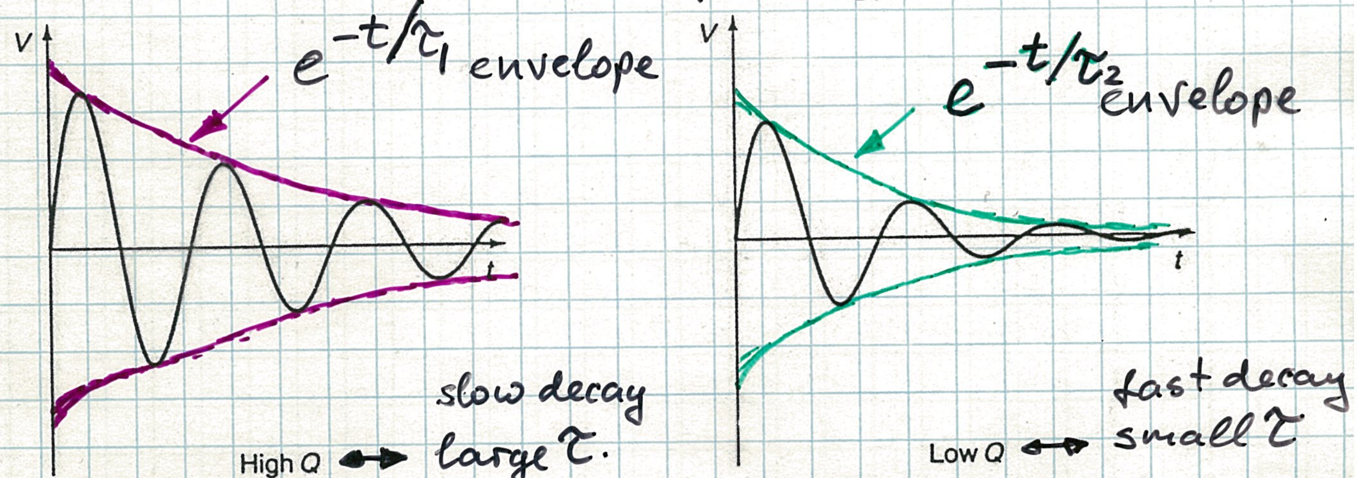


FIGURE 10-3 The effect of Q on the lifetime of a decaying oscillation.

Recall:

$$\alpha_{1,2} = -\frac{R}{2L} \pm \cdots \quad \Rightarrow \quad \tau \sim \frac{L}{R} \sim \frac{1}{\Delta\omega}$$

- Calculate voltages across components on-resonance:

$$V_R = IR = V_0 \cos\omega_0 t \qquad \equiv \text{source voltage}$$

$$\left.\begin{aligned} V_C &= \frac{1}{C}\int I\,dt = \frac{\omega_0 L}{R} V_0 \sin\omega_0 t \\ V_L &= L\frac{dI}{dt} = -\frac{\omega_0 L}{R} V_0 \sin\omega_0 t \end{aligned}\right\} \text{Equal, opposite, and } 90^\circ \text{ out-of-phase with source}$$

NB: $|V_L|, |V_C| = Q V_0 \leftarrow$ can be $\gg V_0$!

resonant high-Q circuit $\rightarrow$ HIGH VOLTAGES!

Ex. 1. radio antenna circuit.

High Q = high frequency selectivity

= high sensitivity (gain)

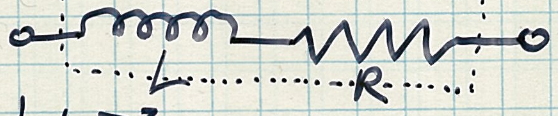
Ex. 2. speaker system.

Avoid resonances to provide broad frequency coverage $\Rightarrow$ Low Q

- Another definition for Q:

$$Q = \frac{\text{max energy stored}}{\text{energy dissipated per radian}}$$

In practice, L =



– max stored $E = \frac{1}{2} L I_{max}^2$

– E dissipated / radian $= \dfrac{I_{av}^2 R}{\omega} = \dfrac{I_{max}^2 R}{2\omega}$

$\Rightarrow Q = \dfrac{\omega L}{R}$, as before.