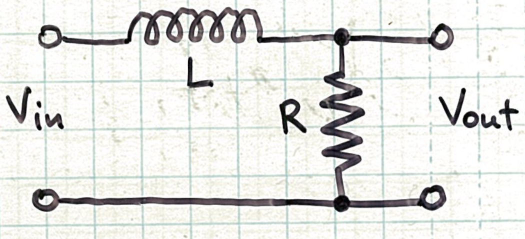
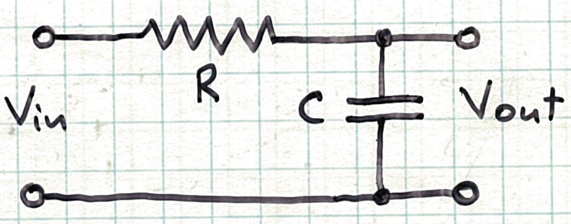
Filter circuits

Filter: pass certain frequencies, reject others

Low-pass filters





Examples of AC voltage dividers:

- RL case: $$V_{out} = \frac{R}{R + j\omega L} V_{in} = \frac{R^2 - j\omega RL}{R^2 + \omega^2 L^2} V_{in}$$

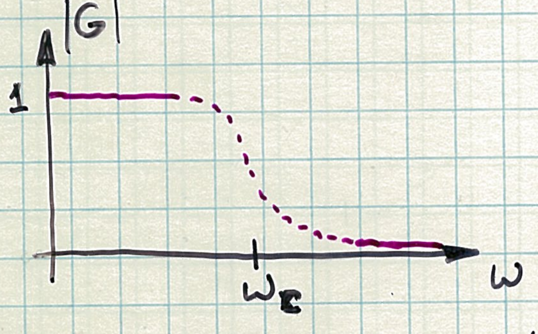
$$\Rightarrow \left| \frac{V_{out}}{V_{in}} \right| = \frac{R}{\sqrt{R^2 + \omega^2 L^2}} = \frac{1}{\sqrt{1 + \left(\frac{\omega L}{R}\right)^2}} = \frac{1}{\sqrt{1 + (\omega/\omega_c)^2}}$$

where $\boxed{\omega_c \equiv \frac{R}{L}}$

Low ω, $\omega \ll \omega_c$ $\Rightarrow$ no attenuation, $V_{out} = V_{in}$

High ω, $\omega \gg \omega_c$ $\Rightarrow$ $V_{out} \to \emptyset$ LOW-PASS

$\omega = \omega_c$, $\left| \frac{V_{out}}{V_{in}} \right| = \frac{1}{\sqrt{2}}$ ← half-power point



ω_c = angular cutoff frequency

$$G \equiv \frac{V_{out}}{V_{in}} = \text{gain}$$

- RC case: $$|G| = \frac{1}{\sqrt{1 + (\omega RC)^2}} = \frac{1}{\sqrt{1 + (\omega/\omega_c)^2}}$$ with $\boxed{\omega_c = \frac{1}{RC}}$

In more detail:

- **AMPLITUDE**: $\left|\frac{V_{out}}{V_{in}}\right| = |G| =$ "attenuation"

can be expressed in decibels:

$$\boxed{A_{dB} \equiv -20 \log_{10}\left|\frac{V_{out}}{V_{in}}\right| = -10 \log_{10}\left|\frac{P_{out}}{P_{in}}\right|}$$ def-n.

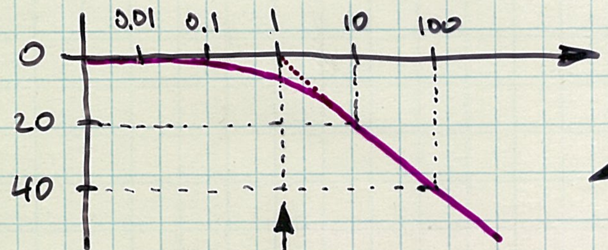
Interpret: $A_{dB} = 10 \rightarrow$ power ($\sim V^2$) delivered to load is reduced by a factor 10

$= 20 \rightarrow$ ... by a factor of 100

Low-pass filter:

$$A_{dB} = -10 \log_{10}\left(\frac{1}{1+\omega^2/\omega_c^2}\right) \approx -20 \log_{10}\frac{\omega_c}{\omega}, \quad \omega \gg \omega_c$$

$$= -20\left(\log_{10}\omega_c - \log_{10}\omega\right)$$



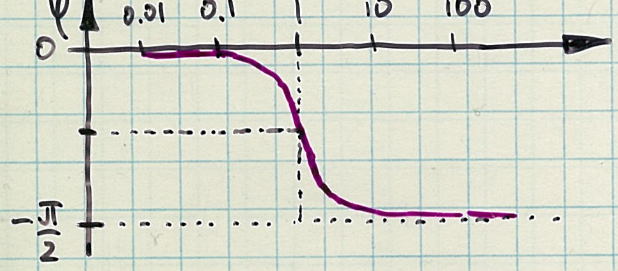
drop-off 20 dB/decade

$\omega = \omega_c \Rightarrow A_{dB} = 20 \log_{10}\sqrt{2} \approx 3\,dB \leftarrow$ "3 dB point"

- **PHASE** $G = \frac{V_{out}}{V_{in}} = \frac{R^2 - j\omega RL}{R^2+\omega^2L^2} \Rightarrow \varphi = \arctan\left(\frac{-\omega L}{R}\right)$

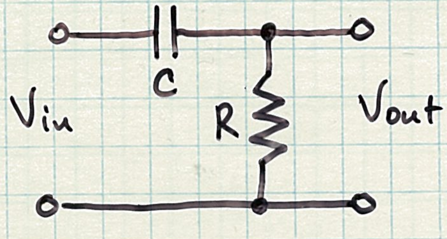
$\varphi = \arctan \frac{-\omega}{\omega_c}$

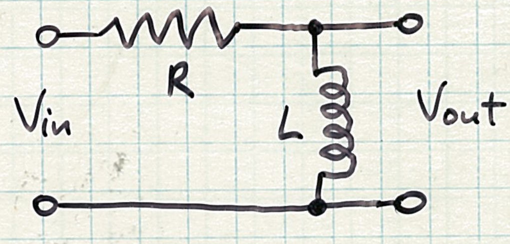
true in both RL and RC circuits



At $\omega = \omega_c$: $\varphi = -45°$

High-pass filters





- **RC case**:

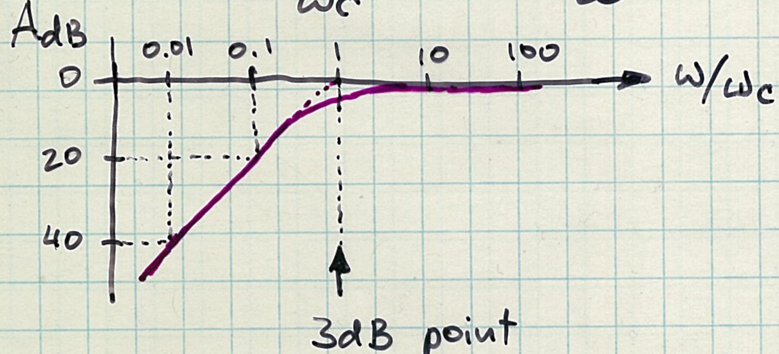
$$V_{out} = \frac{R}{R + \frac{1}{j\omega C}} V_{in} = \frac{\omega^2 R^2 C^2 + j\omega RC}{1 + \omega^2 R^2 C^2} V_{in}$$

$$\left| \frac{V_{out}}{V_{in}} \right| = \frac{\omega RC}{\sqrt{1 + \omega^2 R^2 C^2}} = \frac{1}{\sqrt{1 + \omega_c^2/\omega^2}}, \quad \omega_c = \frac{1}{RC}$$

- **RL case**: same, with $\omega_c = \frac{R}{L}$

$\Rightarrow$ the reverse of the low-pass case:

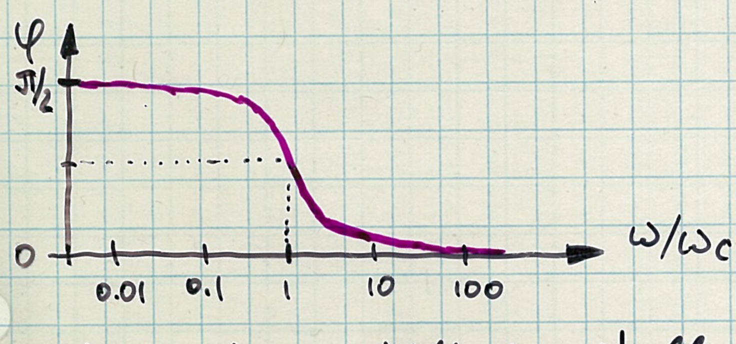
$$\frac{\omega}{\omega_c} \rightarrow \frac{\omega_c}{\omega}$$



$\omega \ll \omega_c$: A increases by 20dB per decade

$\omega \gg \omega_c$: $|G| \rightarrow 1$ $(A = \emptyset$dB

$\omega = \omega_c$: $A = -20 \log_{10} \frac{1}{\sqrt{2}} \approx 3$dB



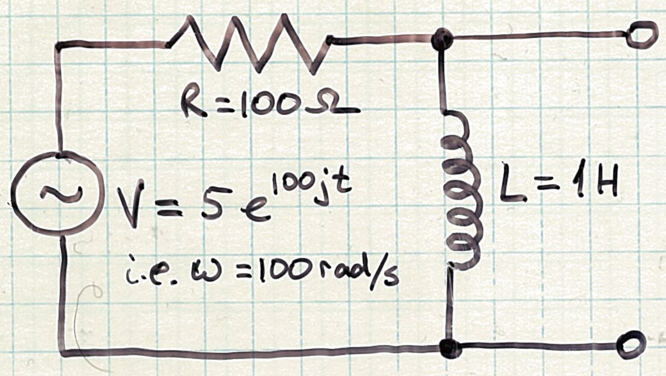
$$\varphi = \arctan \frac{\omega_c}{\omega}$$

$\varphi = +45°$ at $\omega = \omega_c$

Note 1. phase shift inevitable if att-n varies with frequency

Note 2. independent filters in series: attenuations (dB) ADD

Ex. Generalized Thévenin's equivalents



$L = 1H$ @ $\omega = 100$ rad/s

$\Downarrow$

$Z_L = 100j\ \Omega$

A voltage divider:

$$V_L = 5e^{100jt}\,\frac{100j}{100+100j} = 5e^{100jt}\,\frac{j}{1+j}\,\frac{1-j}{1-j} = \frac{1+j}{2}\,5e^{100jt}$$

Magnitude: $\left|\frac{5}{2}(1+j)\right| = \frac{5}{\sqrt{2}} = 3.54$

Phase: $\varphi = \arctan 1 = 45° = \frac{\pi}{4}$

$\Rightarrow$ $V_{o.c.} = V_L = 3.54\,e^{j\pi/4}\,e^{100jt} = 3.54\,e^{j(100t+\pi/4)}$

Thévenin's equivalent: $V_{Th.} = V_{o.c.}$

$$I_{s.c.} = \frac{V}{R} = \frac{5e^{100jt}}{100}$$

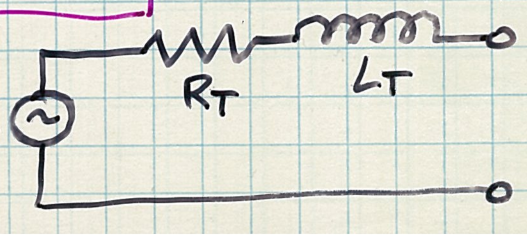
L is shorted out

$$\Rightarrow Z_{Th.} = \frac{V_{o.c.}}{I_{s.c.}} = \frac{\frac{1+j}{2}\,5e^{100jt}}{\frac{5e^{100jt}}{100}} = 100\,\frac{1+j}{2} = 50+50j\ \Omega$$

Represent Z_{Th} as $R_T + j\omega L_T$: @ $\omega = 100$

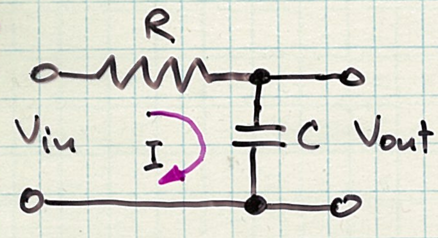
$\boxed{R_T = 50\,\Omega}$ $\boxed{L_T = 0.5H}$

$\Rightarrow$ Equivalent circuit:



$\boxed{V_T = 3.54\,e^{j(100t+\pi/4)}}$

Integrators & differentiators – another use of RC circui[ts]

- 

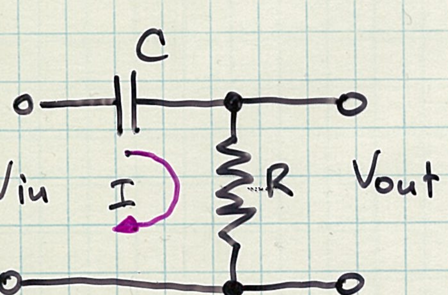
KCL : $\frac{V_{in} - V_{out}}{R} = C \frac{dV_c}{dt}$ $V_c = V_{out}$

if $V_{out} \ll V_{in}$: $\boxed{V_{out} \simeq \frac{1}{RC} \int V_{in}\, dt}$!

RC integrator.

Note : RL integrator is also possible, but capacitors are smaller, cheaper, and more ideal

Note : Vout small is equivalent to t small compared to the time constant of the circuit.

- 

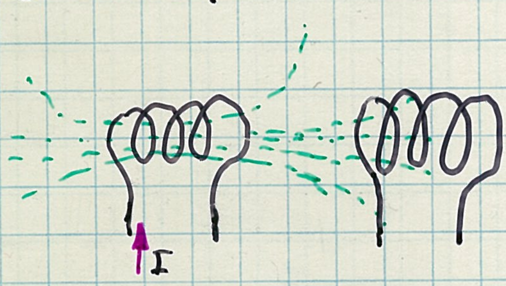
KCL : $C \frac{d}{dt}(V_{in} - V_{out}) = \frac{V_{out}}{R}$

$V_{out} \ll V_{in} \Rightarrow \boxed{V_{out} \simeq RC \frac{dV_{in}}{dt}}$

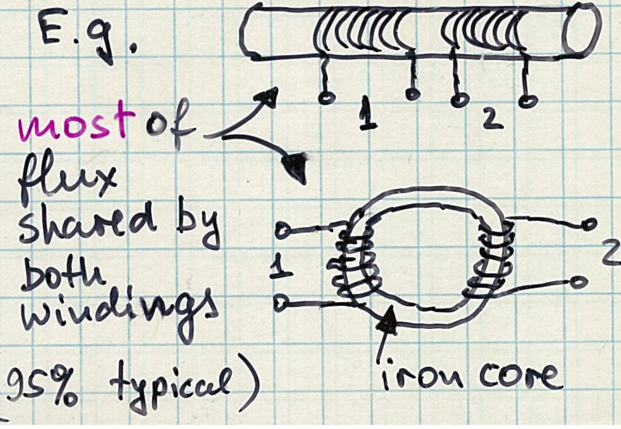
RC differentiator

Transformers

= two inductors placed close enough so that one intercepts [some of] the magnetic flux of the other:



E.g.



Flux in primary $\Phi = \int \frac{V_1}{N_1} dt$ ← turns of primary

Voltage on secondary (Faraday's Law):

$$V_2 = N_2 \frac{d\Phi}{dt}, \quad N_2 = \text{turns of secondary}$$

$$\Rightarrow \boxed{\frac{V_2}{V_1} = \frac{N_2}{N_1}}$$

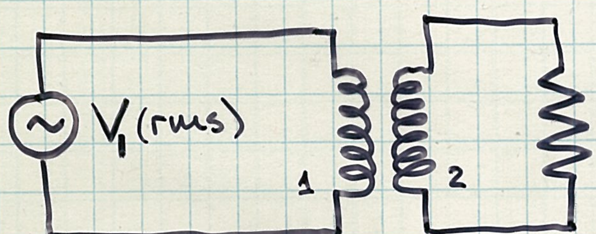
"step-up", $N_2 > N_1$

"step-down", $N_2 < N_1$

Voltage ratio = turns ratio, <u>independent</u> of frequency for ideal transformers.

Real transformers: $I_1 = \frac{V_1}{j\omega L_1}$ ← this limits the low frequency.

<u>Power transfer</u>



rms voltage across R_L is $V_2 = \frac{N_2}{N_1} V_1$

$\Rightarrow$ Power dissipated by R_L: $P = \frac{V_2^2}{R_L} = \frac{N_2^2 V_1^2}{N_1^2 R_L}$

Ideal transformer does not dissipate power, so the current in primary must be:

$$I_1 = \frac{P}{V_1} = \frac{N_2^2 V_1}{N_1^2 R_L} = \frac{V_1}{\frac{N_1^2}{N_2^2} R_L} = \frac{V_1}{R_{eq}}, \quad \boxed{R_{eq} = \frac{N_1^2}{N_2^2} R_L}$$

i.e. the source thinks it is connected to R_{eq}.

Generalize: also true for $\boxed{Z_{eq} = \frac{N_1^2}{N_2^2} Z_L}$

To maximize power transfer, use a transformer to match impedances $Z_{source} = Z_{eq}$