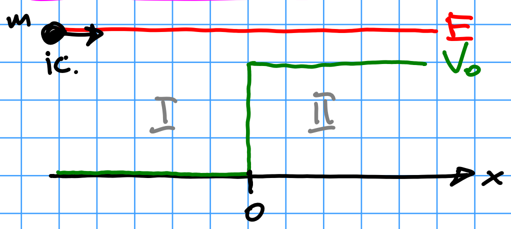
Q.M. wave propagation throug a potential



Classically: free particle will travel left-to-right, kinetic energy lower where $V_0 \neq 0$: $K = E - V_0$

Q.M. $\psi_I = \underbrace{A\, e^{i k_I x}}_{\text{incident}} + \underbrace{B\, e^{-i k_I x}}_{\text{reflected}}$ $\qquad k_I = \dfrac{E}{\hbar^2/2m}$

$\psi_{II} = \underbrace{C\, e^{i k_{II} x}}_{\text{transmitted}} + \cancel{D\, e^{-i k_{II} x}}$ exclude by I.C. incident from II to I $\qquad k_{II} = \dfrac{E - V_0}{\hbar^2/2m}$

Match at the boundary, $x=0$:

$$\left.\begin{array}{lll} \psi_I = \psi_{II} & \longrightarrow & A + B = C \\ \psi_I' = \psi_{II}' & \longrightarrow & k_I (A - B) = k_{II} C \end{array}\right\} \text{ solve} \qquad \frac{C}{A} = \frac{2 k_I}{k_I + k_{II}} \qquad \frac{B}{A} = \frac{k_I - k_{II}}{k_I + k_{II}}$$

Current density $\vec{J} \equiv -\dfrac{i\hbar}{2m}\left(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right)$

so, for example, $J_c = C^* e^{-ik_{II}x}(ik_{II})\, C e^{ik_{II}x} - C e^{ik_{II}x} C^* (-ik_{II})\, e^{-ik_{II}x}$
$= 2 i k_{II} |C|^2$

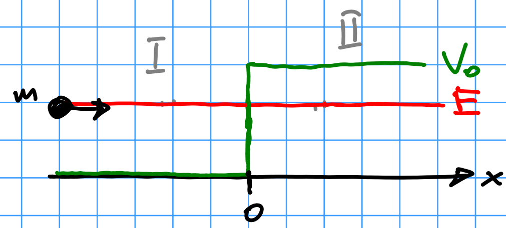
$\Rightarrow$ transmission & reflection coefficients are:

$$T \equiv \frac{J_C}{J_A} = \frac{k_{II} |C|^2}{k_I |A|^2} = \frac{k_{II}\, 4k_I^2}{k_I (k_I + k_{II})^2} = \frac{4 k_I k_{II}}{(k_I + k_{II})^2}$$

$$R \equiv \frac{J_B}{J_A} = \frac{|B|^2}{|A|^2} = \left(\frac{k_I - k_{II}}{k_I + k_{II}}\right)^2$$

EFTS: check that $T + R = 1$

Note: R is the same for $k_I > k_{II}$ and $k_{II} > k_I$
i.e. for step-up and step-down barriers



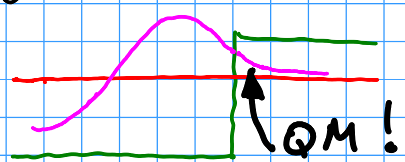
Classically, the particle is confined to I, i.e. it bounces back from the wall.

Q.M: same as before, but since $E - V_0 < 0 \Rightarrow k_{II} \to i k_{II}$

$$\Rightarrow \quad \frac{C}{A} = \frac{2k_I}{k_I + i k_{II}} \quad \text{and} \quad \frac{B}{A} = \frac{k_I - i k_{II}}{k_I + i k_{II}}$$

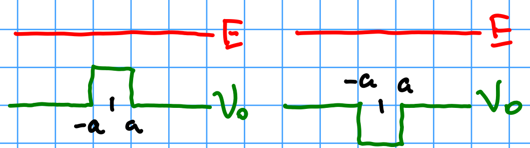
$$\Rightarrow \quad R = \left(\frac{k_I - i k_{II}}{k_I + i k_{II}}\right)^2 = \frac{k_I^2 + k_{II}^2}{k_I^2 + k_{II}^2} = 1$$ i.e. also bounces back

but this does not mean $\psi_{II} = 0$!



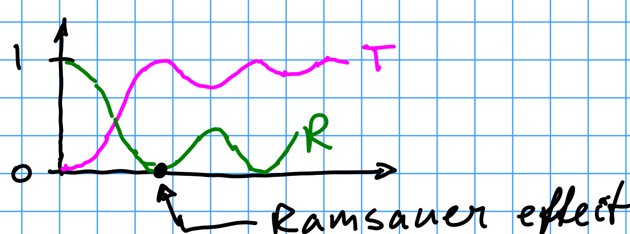
this apparent transparency of barriers is a purely Q.M. effect.

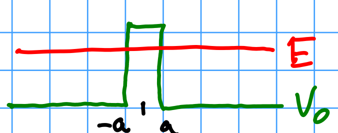
Ex :



$\underline{E > V_0}$: classically, a free particle

Q.M. $\quad \frac{1}{T} = 1 + \frac{1}{4} \frac{V_0^2}{E(E-V_0)} \sin^2 2k_{II} a$





Classically: an impenetrable barrier

Q.M.:
$$\frac{1}{T} = 1 + \frac{1}{4} \frac{V_0^2}{E(V_0 - E)} \sinh^2 2k_{II} a$$

- momentum wavefunctions

a FT pair
$$\begin{cases} \psi(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int \varphi(k)\, e^{ikx}\, dk & \leftarrow \text{coordinate} \\ \varphi(k) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int \psi(x)\, e^{-ikx}\, dx & \leftarrow \text{momentum} \end{cases}$$
representations of the same quantum state

$$\int |\psi(x)|^2 dx = \int |\varphi(k)|^2 dk = 1$$ $\leftarrow$ normalization (Parseval's theorem)

Interpretation: $|\varphi(k)|^2 dk$ = probability that momentum is between $\hbar k$ and $\hbar(k + dk)$

Ex a Gaussian wave "packet":

$$\varphi(k) = \frac{(2\Delta)^{1/2}}{(2\pi)^{1/4}} e^{-(k-k_0)^2\Delta^2}$$

$$\Rightarrow \psi(x,0) = (2\pi\Delta^2)^{-1/4} e^{-x^2/4\Delta^2 + ik_0x}$$

and

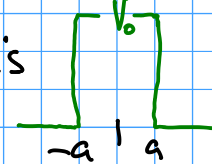
$$\psi(x,t) = (2\pi)^{-1/2}\int \varphi(k)\, e^{-i\omega t} e^{ikx}\, dk$$

$$= (2\pi)^{-1/4}\left(\Delta + \frac{i\hbar t}{2m\Delta}\right)^{-1/2} \exp\left[\frac{-x^2/4\Delta^2 + ik_0x - i\hbar\frac{k_0^2}{2m}t}{1 + i\hbar\frac{t}{2m\Delta^2}}\right]$$

EFTS1: verify this expression

EFTS2: introduce natural scales to make the above dimensionless, i.e. suitable for numeric calculations

e.g. $x \to x' = \frac{x}{a}$ where V_0 barrier potential is



Hint: need ε = energy scale so can choose, e.g.

$V_0 = 16\,\varepsilon$, $E = 4\varepsilon = \frac{1}{4}V_0$ etc.

one candidate: $\varepsilon = \frac{\hbar^2}{2m}\frac{1}{a^2}$ has dimensions of energy:

$$[\varepsilon] = \frac{(Js)^2}{kg\ m^2} = \frac{kg^2\, m^4}{s^2\, kg\, m^2} = \frac{kg\, m^2}{s^2} = J$$