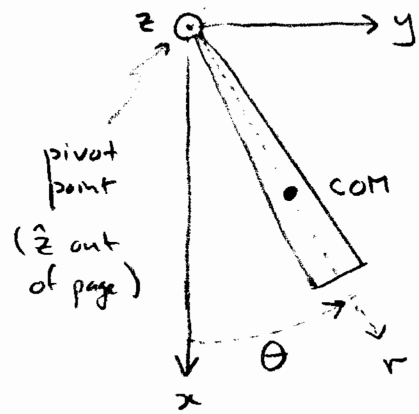
CAP 2005

University Prize Exam

Solutions

Problem 1: Mechanics (Pendulum)



Given: L = length of (thin) rod

$\rho(r)$ = linear mass density

$= \rho_0 \frac{r}{L}$

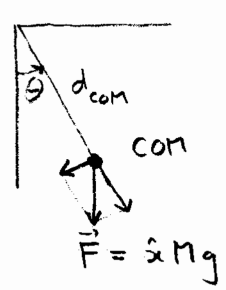
(a) M = total mass of rod $= \int_0^L \rho(r)\,dr = \frac{\rho_0}{L}\frac{r^2}{2}\Big|_0^L = \frac{\rho_0 L}{2}$

$d_{com} = \frac{1}{M}\int_0^L r\,\rho(r)\,dr = \frac{1}{M}\frac{\rho_0}{L}\frac{r^3}{3}\Big|_0^L = \frac{2}{\rho_0 L}\frac{\rho_0 L^2}{3} = \frac{2}{3}L$

(b) I = moment of inertia about pivot point

$= \int_0^L r^2 \rho(r)\,dr = \frac{\rho_0}{L}\frac{r^4}{4}\Big|_0^L = \frac{\rho_0 L^3}{4}$

(c)

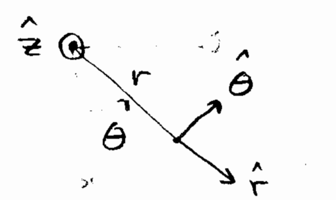


$\vec{\tau}$ = torque

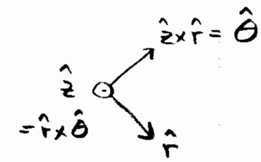
$= \vec{r}\times\vec{F}$

$= \hat{r}\, d_{com} \times (\hat{r}\cos\theta - \hat{\theta}\sin\theta)Mg$

$= -\hat{z}\, d_{com} Mg \sin\theta$



$\hat{x} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$



(d) $\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d}{dt}\left\{\sum_i \vec{r}_i \times \vec{p}_i\right\} = \frac{d}{dt}\left\{\sum_i m_i \vec{r}_i \times \vec{v}_i\right\}$

$= \frac{d}{dt}\left\{\sum_i m_i \vec{r}_i \times (\vec{\omega}\times\vec{r}_i)\right\} = \frac{d}{dt}\int_0^L dr\,\rho(r)\,\hat{r}r \times \left(\hat{z}\frac{d\theta}{dt}\times\hat{r}r\right)$

$= \frac{d}{dt}\left\{\hat{z}\int_0^L dr\,\rho(r)\,r^2\frac{d\theta(t)}{dt}\right\} = \frac{d}{dt}\left\{\hat{z}\,I\dot{\theta}\right\} = \hat{z}\,I\ddot{\theta}$

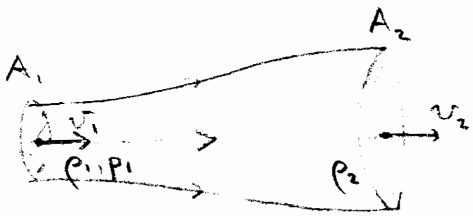
$\therefore\; I\ddot{\theta} = -d_{com}Mg\sin\theta \approx -d_{com}Mg\,\theta$ for small θ

$\ddot{\theta} \approx -\omega_0^2\,\theta$

where $\omega_0 = \sqrt{\frac{d_{com}Mg}{I}} = \sqrt{\frac{8Mg}{3\rho_0 L^2}}$ ← check units: $\frac{1}{sec.}$ ✓

Problem 2: Fluid Flow (Windmill)

Concepts: For horizontal fluid flow (gravity not a factor):



A = cross-sectional area of bundle of streamlines

v = fluid velocity

ρ = fluid density

p = fluid pressure

(1) Conservation of <u>mass</u> $\Rightarrow$ $\rho_1 A_1 v_1 = \rho_2 A_2 v_2$

Incompressible $\Rightarrow$ $\rho = \text{const.}$ $\Rightarrow$ $A_1 v_1 = A_2 v_2$

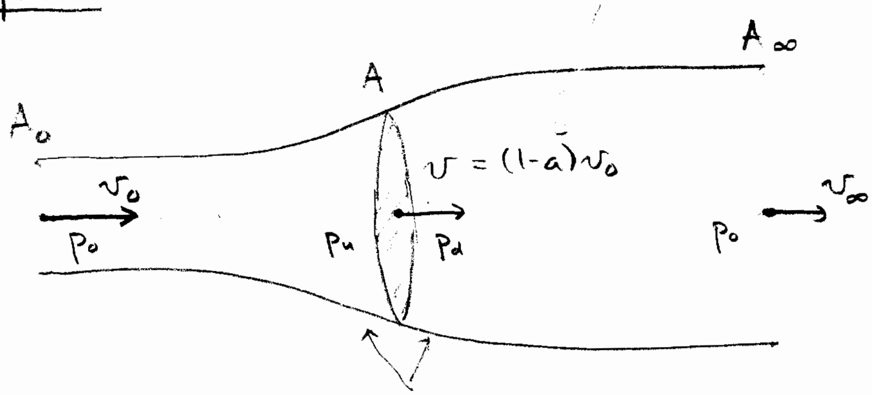
<u>i.e.</u> $\boxed{Av = \text{const. along streamlines}}$

(2) Conservation of <u>energy</u> $\Rightarrow$ $p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$

<u>i.e.</u> $\boxed{p + \frac{1}{2}\rho v^2 = \text{const. along streamlines}}$ (Bernoulli)

unknown factor "a" $(0 < a < 1)$

<u>The problem:</u>

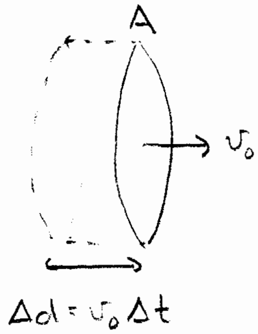


$\rho = \text{const.}$ (incompressible)

velocity continuous across windmill

pressure <u>discontinuous</u> " "

(a) If air were flowing at a speed v_0 through the windmill of area A [rather than its actual speed $v = (1-a)v_0$], at what rate would KE be transported through A ?



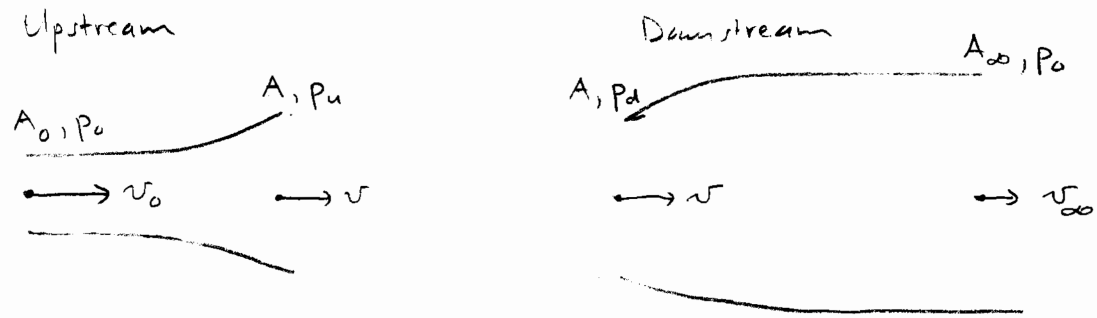
During a time Δt, the air in the volume $\Delta V = A \Delta d$ behind A will pass through A, when $\Delta d = v_0 \Delta t$. The mass of this cylinder of air is $\Delta m = \rho \Delta V$, moving at speed v_0

$$\therefore \Delta KE = \frac{1}{2} \Delta m v_0^2 = \frac{1}{2} (\rho \Delta V) v_0^2 = \frac{1}{2} \rho A v_0^3 \Delta t$$

i.e. $P_N \equiv$ "Power in undisturbed wind passing through area A"

$$= \frac{\Delta KE}{\Delta t} = \frac{1}{2} \rho A v_0^3$$

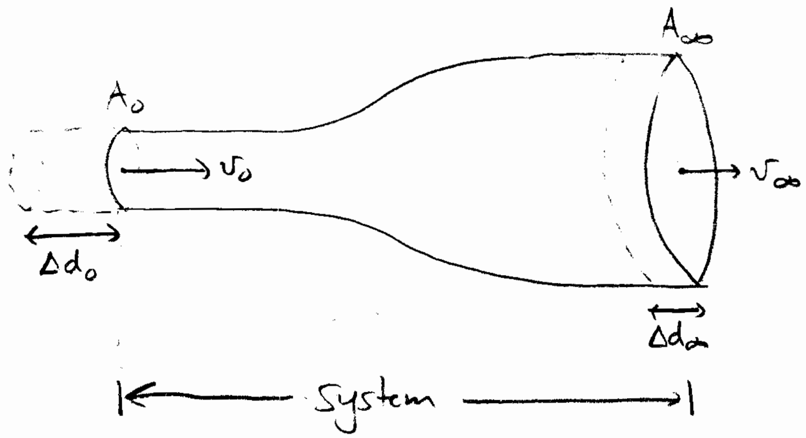
(b) Bernoulli's eq'n can be applied upstream and downstream of A, but NOT across A (the assumptions of "nice" fluid flow break down near the windmill itself)



$$p_0 + \frac{1}{2} \rho v_0^2 = p_u + \frac{1}{2} \rho v^2 \qquad p_d + \frac{1}{2} \rho v^2 = p_0 + \frac{1}{2} \rho v_\infty^2$$

Subtract : $\boxed{\Delta p \equiv p_u - p_d = \frac{1}{2} \rho (v_0^2 - v_\infty^2)}$

(c)



During a time Δt, the volume of air on the left ($\Delta d_0 = v_0 \Delta t$) enters the system, and the one on the right ($\Delta d_\infty = v_\infty \Delta t$) leaves the system.

$$\begin{aligned} \therefore \quad \Delta p &= \text{change in momentum of system} \\ &= (\text{momentum added}) - (\text{momentum removed}) \\ &= \Delta m_0 v_0 - \Delta m_\infty v_\infty \\ &= (\rho \Delta V_0) v_0 - (\rho \Delta V_\infty) v_\infty \\ &= (\rho A_0 \Delta d_0) v_0 - (\rho A_\infty \Delta d_\infty) v_\infty \\ &= \rho A_0 v_0^2 \Delta t - \rho A_\infty v_\infty^2 \Delta t \\ &= \rho (A_0 v_0^2 - A_\infty v_\infty^2) \Delta t \end{aligned}$$

By Newton's 2nd Law:

$\Delta p = F \Delta t$ = impulse imparted to windmill during time Δt (in order to conserve momentum)

(Think of windmill as a "permeable disk" of area A, letting the wind through, but feeling a force exerted on it. This force is in the direction of the wind, which is directed "sideways" by tilting the vanes of the windmill, resulting in a torque on windmill making it rotate and do useful work)

$$\therefore \quad \boxed{F = \frac{\Delta p}{\Delta t} = \rho (A_0 v_0^2 - A_\infty v_\infty^2)}$$

(d) The force F can also be thought of as due to the difference in pressure on the upwind and downwind sides of the windmill:

$$F = A\,\Delta p = \frac{1}{2}\rho A (v_0^2 - v_\infty^2) \quad \text{from part (b)}$$

$$\therefore \quad \cancel{\rho}(A_0 v_0^2 - A_\infty v_\infty^2) = \frac{1}{2}\cancel{\rho} A (v_0^2 - v_\infty^2)$$

use $Av = \text{const}$ (conservation of mass)

$$\Rightarrow \quad A_\infty v_\infty^2 = (A_\infty v_\infty) v_\infty = (A_0 v_0) v_\infty$$

$$\therefore \quad A_0 v_0 (v_0 - v_\infty) = \frac{A}{2}(v_0 + v_\infty)\cancel{(v_0 - v_\infty)}$$

$$\Rightarrow \quad Av = A_0 v_0 = \frac{A}{2}(v_0 + v_\infty)$$

i.e. $\boxed{v = \frac{1}{2}(v_0 + v_\infty)}$ (average of v_0 and v_∞)

we set $v = (1-a)v_0$ (a an unknown factor)

$$\therefore \quad (1-a)v_0 = \frac{1}{2}(v_0 + v_\infty)$$

i.e. $\boxed{v_\infty = (1-2a)v_0}$ $\left(\Rightarrow \underset{v_\infty = v_0}{0} < a < \underset{v_\infty = 0}{\frac{1}{2}}\right)$

$P \equiv$ power extracted by windmill

= (rate of flow of KE far upstream) − (rate of flow of KE far downstream)

... from part (a): $$\frac{\Delta KE}{\Delta t} = \begin{cases} \frac{1}{2}\rho A_0 v_0^3 & \text{far upstream} \\ \frac{1}{2}\rho A_\infty v_\infty^3 & \text{far downstream} \end{cases}$$

$$\therefore \quad P = \frac{1}{2}\rho (A_0 v_0^3 - A_\infty v_\infty^3)$$

With an eye towards $P_N = \frac{1}{2}\rho A v_0^3$ in part (a), rewrite this as:

$$P = \frac{1}{2}\rho A v_0^3 \frac{1}{A v_0^3}\left(A_0 v_0^3 - A_\infty v_\infty^3\right)$$

$$= P_N\left(\frac{A_0}{A} - \frac{A_\infty}{A}\frac{v_\infty^3}{v_0^3}\right) \quad , \text{ use } A_\infty v_\infty = A_0 v_0$$

$$= P_N\left(\frac{A_0}{A} - \frac{A_0 v_0}{A}\frac{v_\infty^2}{v_0^3}\right)$$

$$= P_N \frac{A_0}{A}\left(1 - \frac{v_\infty^2}{v_0^2}\right)$$

We know: $1 - \frac{v_\infty^2}{v_0^2} = 1 - (1-2a)^2 = 4a(1-a)$

And: $A_0 v_0 = A v \implies \frac{A_0}{A} = \frac{v}{v_0} = \frac{1}{2}\left(1 + \frac{v_\infty}{v_0}\right) = (1-a)$

$$\therefore \boxed{P = 4a(1-a)^2 P_N} \qquad 0 < a < \frac{1}{2}$$

(e) $P(a)$ max where $\frac{dP}{da} = 0$

$$0 = \frac{d}{da}\left[4a(1-a)^2\right] = 4(1-a)^2 - 8a(1-a) = 4(1-a)(1-3a)$$

sol'n $a = 1$ not allowed $\left(0 < a < \frac{1}{2}\right)$

$$\implies \boxed{a = \frac{1}{3}}$$

$$4\left(\frac{1}{3}\right)\left(1 - \frac{1}{3}\right)^2 = \frac{4}{3}\frac{4}{9} = \frac{16}{27}$$

$$\boxed{P_{max} = \frac{16}{27} P_N}$$

... a little over $\frac{1}{2}$ the "raw wind power" can be extracted under these idealized assumptions

Problem 3 : Thermodynamics / Heat Capacity

Concepts : (1) Conservation of energy :

$$dU = \delta Q - \delta W$$

(Change in internal energy) = (heat added) − (work done)

(2) $dS = \dfrac{\delta Q}{T}$ (for an infinitesimal reversible transformation)

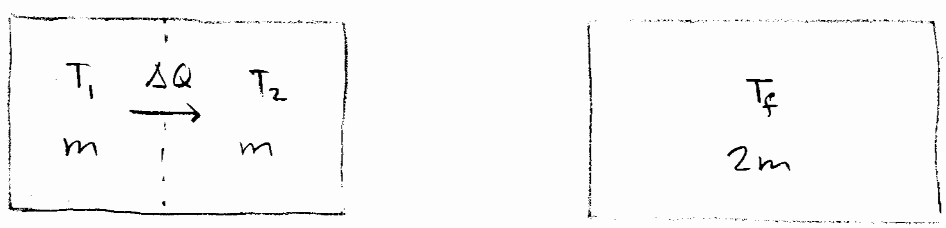
$\dfrac{1}{\text{Temp.}}$ is the integrating factor converting δQ into an exact differential

(there is no such thing as a "state function Q", only the "process" δQ)

(3) $\displaystyle\int_A^B \frac{\delta Q}{T} \leq S(B) - S(A)$ (equality for reversible transformation)

(4) Heat capacity $C_p = \left(\dfrac{\partial U}{\partial T}\right)_{P\ \text{const.}}$ [or C_V with V const.]

The Problem :



(P = const.)

initial state $(T_1 > T_2)$ final state $(T_1 > T_f > T_2)$

in general : $\displaystyle \Delta S \geq \int_i^f \frac{\delta Q}{T} = \int_i^f \frac{dU + \delta W}{T} \underset{\uparrow}{=} \int_i^f \frac{dU}{T}$

when $\delta W = 0$
(no work done)

$$dU = \underset{\substack{\uparrow \\ \text{heat capacity}}}{C_p}\, dT = m \underset{\substack{\uparrow \\ \text{specific heat (heat cap. / unit mass)}}}{c_p}\, dT \quad \text{(at constant pressure)}$$

$$\therefore \quad \Delta S_1 \geq \int_i^f \frac{dU_1}{T} = m c_p \int_i^f \frac{dT}{T} = m c_p \ln\left(\frac{T_f}{T_1}\right)$$

$$\Delta S_2 \geq \int_i^f \frac{dU_2}{T} = m c_p \int_i^f \frac{dT}{T} = m c_p \ln\left(\frac{T_f}{T_2}\right)$$

$$\therefore \quad \Delta S = \Delta S_1 + \Delta S_2 \geq m c_p \ln\left(\frac{T_f^2}{T_1 T_2}\right) = 2 m c_p \ln\left(\frac{T_f}{\sqrt{T_1 T_2}}\right)$$

- What is T_f ?

$$\Delta U_1 = \int_i^f dU_1 = mc_p \int_i^f dT = mc_p (T_f - T_1)$$
$$\Delta U_2 = \int_i^f dU_2 = mc_p \int_i^f dT = mc_p (T_f - T_2)$$

$$\therefore \quad \Delta U = \Delta U_1 + \Delta U_2 = mc_p \left[2T_f - (T_1 + T_2) \right]$$

__but__: For the system __as a whole__, no heat is added (adiabatic) and no work is done

$$\Rightarrow \quad \Delta U = 0$$

$$\Rightarrow \quad \boxed{T_f = \frac{1}{2}(T_1 + T_2)}$$

- Let $\Delta S_{min} = 2mc_p \ln\left(\frac{T_f}{\sqrt{T_1 T_2}}\right)$ (if transf. __reversible__)

is $\Delta S_{min} \geq 0$? (In which case, $\Delta S > 0$)

$$\Leftrightarrow \quad \text{is} \quad \frac{T_f}{\sqrt{T_1 T_2}} \geq 1 \quad ?$$

$$\Leftrightarrow \quad T_f^2 \geq T_1 T_2 \quad ?$$

$$\Leftrightarrow \quad \frac{1}{4}(T_1 + T_2)^2 \geq T_1 T_2 \quad ?$$

$$\Leftrightarrow \quad (T_1 - T_2)^2 \geq 0 \quad ?$$

obviously true ✓

(equality when $T_1 = T_2$, __i.e.__ initial state = final state ... no transf.)

c

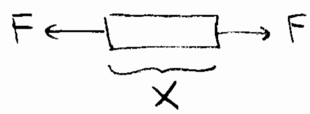
Problem 4: Classical Statistical Mechanics (Spring)

Concepts: (1) Entropy $S = k \ln \Gamma$ (written on Boltzmann's tombstone!)

where k = Boltzmann's constant

Γ = "number of microstates compatible with a given macrostate"

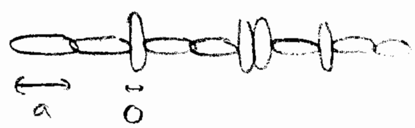
(2) $\dfrac{1}{T} \equiv \dfrac{\partial S(E)}{\partial E}$ (definition of temperature)

(3) 

An elastic body of length X under tension F exhibits Hooke's law if $F \propto X$, i.e. $d^2F/dX^2 = 0$ -- a stationary point of $F(X)$

The Problem:

(a)



n = total number of links

r = number of links that are horizontal (length "a")

$(n-r)$ = " " " " " vertical (length 0)

$\therefore$ length of chain $X = ra = n\left(\frac{r}{n}a\right) \equiv nx$, $x = \frac{r}{n}a$

Macroscopically, the different states of the chain are distinguished only by the total length X, i.e. the value of x, i.e. the value of r. However, if we have a set of n objects, r of one type (Horizontal) and $(n-r)$ of another type (Vertical), there are many ways to construct a sequence of these, i.e. many different orderings of the objects. Each such ordering is a microstate "compatible with" the given macrostate (value of r). The number of possible microstates is Γ. What is Γ?

We have n objects in total $\{r$ of type H, $(n-r)$ of type V $\}$.
Thus we have n objects to choose from for the first object in our sequence.
Then we are left with $(n-1)$ objects to choose from for the second object in our sequence, etc. In total, there are $n(n-1)(n-2)\cdots(2)(1) = n!$ possible sequences. <u>But</u>: not all of these sequences are <u>distinguishable</u>.

Example: $n=5, r=3$: HHVHV is one possible sequence

but if we interchange these two H's, the new sequence is indistinguishable from the original

$\therefore$ we must <u>divide</u> $n!$ by $r!$ (number of ways the H's in a given sequence can be rearranged and still result in the same sequence) and $(n-r)!$ (number of ways the V's in a given sequence can be rearranged and not change the sequence).

<u>i.e.</u> $\Gamma = \dfrac{n!}{r!(n-r)!}$ ← <u>corrects for "overcounting"</u>

$$\therefore \quad S = k \ln\left(\frac{n!}{r!(n-r)!}\right)$$

Assuming all numbers are <u>large</u>, we can use <u>Stirling's approximation</u>:

$$\ln N! \sim N \ln N - 1$$

$$\therefore \quad \frac{S}{k} = \ln n! - \ln r! - \ln(n-r)!$$

$$\sim n \ln n - r \ln r - (n-r)\ln(n-r) + 1$$

use: $r = n\dfrac{x}{a}$, $(n-r) = n\left(1-\dfrac{x}{a}\right)$

$$\frac{S}{k} \sim n \ln n - n\frac{x}{a}\ln\left(n\frac{x}{a}\right) - n\left(1-\frac{x}{a}\right)\ln\left\{n\left(1-\frac{x}{a}\right)\right\} + 1$$

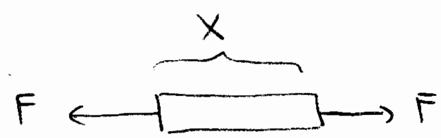
$$= n\ln n - n\frac{x}{a}\ln n - n\frac{x}{a}\ln\frac{x}{a} - n\left(1-\frac{x}{a}\right)\ln n - n\left(1-\frac{x}{a}\right)\ln\left(1-\frac{x}{a}\right)$$

$$= \cancel{n\left(1-\frac{x}{a}\right)\ln n - n\left(1-\frac{x}{a}\right)\ln n} - n\left\{\frac{x}{a}\ln\frac{x}{a} + \left(1-\frac{x}{a}\right)\ln\left(1-\frac{x}{a}\right)\right\} + 1$$

drop this

$$\boxed{S = -nk\left[\frac{x}{a}\ln\frac{x}{a} + \left(1-\frac{x}{a}\right)\ln\left(1-\frac{x}{a}\right)\right]}$$

(b) $\frac{1}{T} \equiv \frac{\partial S}{\partial E}$; need S as a function of E (energy)



To change a link from horizontal to vertical requires the chain to shrink by a length "a", i.e. an amount of work $W =$ force $\times$ distance $= Fa$ to be done. If we take $E = 0$ to correspond to the case when all links are horizontal $(r = n)$, then:

$$\boxed{E = (n-r)\cdot Fa = n\cdot\left(1-\frac{x}{a}\right)\cdot Fa}$$

$r = n \rightarrow E = 0$ (all links horizontal)

$r = 0 \rightarrow E = nFa \equiv E_0$ (max energy — all links vertical)

$$\therefore \left(1-\frac{x}{a}\right) = \frac{E}{E_0}, \quad \frac{x}{a} = 1 - \frac{E}{E_0}$$

$$\boxed{S(E) = -nk\left[\left(1-\frac{E}{E_0}\right)\ln\left(1-\frac{E}{E_0}\right) + \frac{E}{E_0}\ln\frac{E}{E_0}\right]}$$

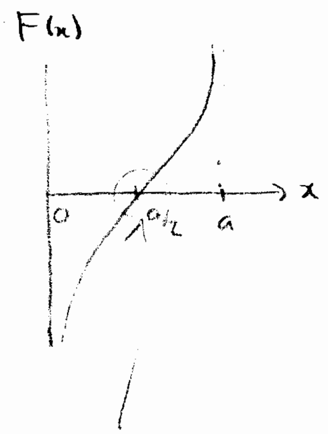
$$\frac{1}{T} = \frac{\partial S}{\partial E} = -nk\left[-\frac{1}{E_0}\ln\left(1-\frac{E}{E_0}\right) - \cancel{\frac{1}{E_0}} + \frac{1}{E_0}\ln\frac{E}{E_0} + \cancel{\frac{1}{E_0}} \right]$$

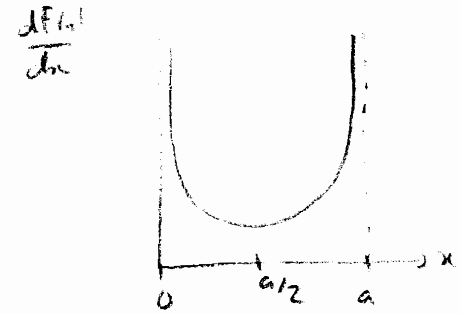
$$= -\frac{nk}{E_0}\ln\left(\frac{E/E_0}{1-E/E_0}\right) = -\frac{k}{Fa}\ln\left(\frac{E/E_0}{1-E/E_0}\right)$$

$$= -\frac{k}{Fa}\ln\left(\frac{1-x/a}{x/a}\right) = -\frac{k}{Fa}\ln\left(\frac{a}{x}-1\right)$$

$$\boxed{F(x) = -\frac{kT}{a}\ln\left(\frac{a}{x}-1\right)}$$

(c)





$F(x)$ is <u>linear</u> in x <u>near</u> $x = a/2$

<u>i.e.</u> $\frac{d^2F(x)}{dx^2} = 0$ (stationary point) at $x = a/2$

$$\frac{dF}{dx} = -\frac{kT}{a}\left(\frac{a}{x}-1\right)^{-1}\left(-\frac{a}{x^2}\right) = \frac{kT}{x(a-x)}$$

$$\frac{d^2F}{dx^2} = -kT\,\frac{(a-2x)}{x^2(a-x)^2} = 0 \quad \text{at} \quad x = \frac{a}{2} \quad ✓$$

set $x = \frac{a}{2} + \xi$, $|\xi| \ll a$

$$F \approx \cancel{F\left(\frac{a}{2}\right)}^{\,0} + \xi \left.\frac{dF}{dx}\right|_{x=a/2} = \frac{kT}{(a/2)^2}\,\xi$$

convert to macroscopic: $X = nx$ & $\Xi = n\xi$ = macroscopic stretch of spring

$F = \kappa\,\Xi$, where $\kappa = \frac{kT}{n(a/2)^2}$ = Hooke's constant for this "spring"

Problem 5 : Lagrangian Mechanics

Concepts : (1) Lagrangian $L(q,\dot{q}) = KE - PE$

(2) Euler-Lagrange equations : $0 = \delta \int dt\, L(q,\dot{q})$

(3) Effective potential

The Problem :

(a) $F_r(r) = -kr^2$ (attractive central force, $k > 0$)

$\vec{F} = -\vec{\nabla} V \Rightarrow V(r) = \frac{1}{3} k r^3 + \underbrace{\text{const.}}_{\equiv 0}$

$PE = V(r)$

$KE = \frac{1}{2} m |\vec{v}|^2 = \frac{1}{2} m (v_r^2 + v_\theta^2) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$

$L(q,\dot{q}) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{1}{3} k r^3$

(b) $0 = \delta \int dt\, L(q,\dot{q}) = \int dt \left\{ \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right\}$

$= \int dt \left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \underbrace{\delta q}_{\delta q |_{\text{endpoints}} \equiv 0} \right) - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \right) \delta q \right\}$

$\Rightarrow 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q}$

$q = r : \quad 0 = \frac{d}{dt}(m\dot{r}) - (mr\dot{\theta}^2 - kr^2) = m(\ddot{r} - r\dot{\theta}^2) + kr^2$ (1)

$q = \theta : \quad 0 = \frac{d}{dt}(mr^2\dot{\theta}) - (0) = mr(r\ddot{\theta} + 2\dot{r}\dot{\theta})$ (2)

(c) Energy : $E = KE + PE = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$

Angular Momentum : $L = \hat{z} \cdot (\vec{r} \times \vec{p}) = \hat{z} \cdot (\hat{r} r) \times m(\hat{r} v_r + \hat{\theta} v_\theta)$

$= m r v_\theta = m r^2 \dot{\theta}$

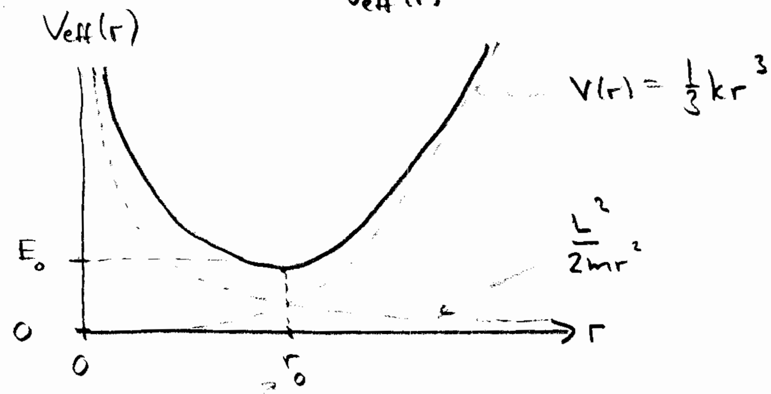
check : $\dot{E} = m(\dot{r}\ddot{r} + r\dot{r}\dot{\theta}^2 + r^2\dot{\theta}\ddot{\theta}) + kr^2\dot{r}$; use $r\ddot{\theta} = -2\dot{r}\dot{\theta}$ by (2)

$= \dot{r}\{ m(\ddot{r} - r\dot{\theta}^2) + kr^2 \} = 0$ by (1) ✓

$\dot{L} = \frac{d}{dt}(mr^2\dot{\theta}) = 0$ by (2) ✓

(d) use $L = mr^2\dot{\theta} = \text{const}$ to write:

$$E = \frac{1}{2}m\dot{r}^2 + \underbrace{\frac{L^2}{2mr^2} + V(r)}_{V_{eff}(r)}$$



radius of stable circular orbit

Find r_0: $\quad 0 = \left.\frac{dV_{eff}}{dr}\right|_{r_0} = \left\{-\frac{L^2}{mr^3} + kr^2\right\}\Big|_{r_0} \Rightarrow \boxed{r_0^5 = \frac{L^2}{km}}$

$$\therefore \quad \left.\frac{d^2V_{eff}}{dr^2}\right|_{r_0} = \left\{\frac{3L^2}{mr^4} + 2kr\right\}_{r_0} = 2kr_0\left\{1 + \frac{3L^2}{2km}\frac{1}{r_0^5}\right\} = 5kr_0$$

For small radial oscillations about the stable circular orbit, set

$r(t) = r_0 + \delta r(t)$, $\delta r(t)$ small

$$E = \frac{1}{2}m\,\delta\dot{r}^2 + V_{eff}(r_0 + \delta r)$$

$$\approx \frac{1}{2}m\,\delta\dot{r}^2 + V_{eff}(r_0) + \cancel{\left.\frac{dV_{eff}}{dr}\right|_{r_0}}^{0}\delta r + \frac{1}{2}\left.\frac{d^2V_{eff}}{dr^2}\right|_{r_0}\delta r^2$$

$$\dot{E} = 0 \approx m\,\delta\dot{r}\,\delta\ddot{r} + \left.\frac{d^2V_{eff}}{dr^2}\right|_{r_0}\delta r\,\delta\dot{r}$$

$$\Rightarrow \quad 0 = m\,\delta\ddot{r} + \left.\frac{d^2V_{eff}}{dr^2}\right|_{r_0}\delta r$$

This equation is of the form $\ddot{x} + \omega_0^2 x = 0$ with oscillation frequency

$$\omega_0 = \sqrt{\frac{1}{m}\left.\frac{d^2V_{eff}}{dr^2}\right|_{r_0}} = \sqrt{\frac{1}{m}5kr_0}\,, \text{ with } r_0 \text{ given above.}$$

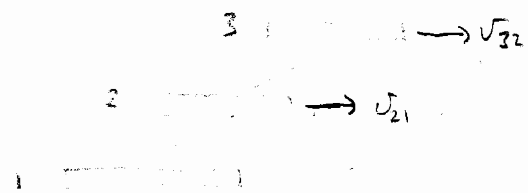
Problem 6 : Atomic Physics & Special Relativity

Concepts : (1) For a hydrogen atom the energy levels are of the form:

$$E_n = -\frac{\mathcal{E}}{n^2}, \quad n = 1, 2, 3, \ldots$$

for some constant $\mathcal{E}$ ($= 13.6$ eV)

(2) Relativistic velocity addition :

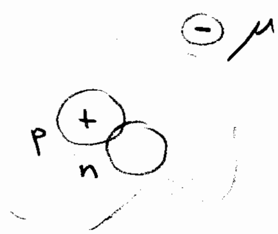


$$v_{31} = \frac{v_{32} + v_{21}}{1 + v_{32} v_{21}/c^2}$$

"3 relative to 1"

The Problem :

(a) The muonic atom (below) differs from the hydrogen atom in that the electron mass (m_e) has been replaced with the muon mass ($m_\mu \approx 207\, m_e$), and that the mass of the nucleus (m_N) has been approximately doubled:



$$m_N = m_p \longrightarrow m_N = m_p + m_n \approx 2 m_p$$

where m_p & m_n are the proton & neutron masses.

$\therefore$ The energy levels of the muonic atom must be of the form $E_n' = -\frac{\mathcal{E}'}{n^2}$ for some constant $\mathcal{E}'$. To determine $\mathcal{E}'$ we need only recall how $\mathcal{E}$ for the hydrogen atom depends on mass :

$$\mathcal{E} = \frac{m_e e^4}{8 \epsilon_0^2 h^2} \propto m_e$$

$$\therefore \mathcal{E}' \approx \frac{m_\mu e^4}{8 \epsilon_0^2 h^2} \approx 207\, \mathcal{E} \approx 207 \times 13.6 \text{ eV} \approx \underline{2815 \text{ eV}}$$

$\therefore$ Ground state energy $E_1' = -\frac{\mathcal{E}'}{1^2} = -2815$ eV

1st excited state energy $E_1' = -\frac{\mathcal{E}'}{2^2} = -704$ eV

Additional notes:

(i) The mass m_e appearing in the above formula for $\mathcal{E}$ is really the **reduced mass** $m_* = \left(\frac{1}{m_e} + \frac{1}{m_N}\right)^{-1}$, which is approximately m_e since $m_N = m_p \approx 1836\, m_e \gg m_e$. For the muonic atom, this approximation is not such a good one, and we should really use the reduced mass:

$$m_* = \left(\frac{1}{m_\mu} + \frac{1}{m_N}\right)^{-1} = \left(\frac{1}{207 m_e} + \frac{1}{2\times 1836\, m_e}\right)^{-1}$$
$$\approx 196\, m_e$$

$$\therefore\ \mathcal{E}' \approx 196 \times 13.6\ \text{eV} = \underline{\underline{2666\ \text{eV}}}$$

... about 5% lower than the value given above

(ii) There is a very simple and quantum-intuitive way to derive the ground state energy of a hydrogen-like atom (up to a numerical factor):

Start with total energy $E = KE + PE$ (kinetic + potential)

If the atom has a "fuzzy", quantum mechanical "size" (radius) R, then

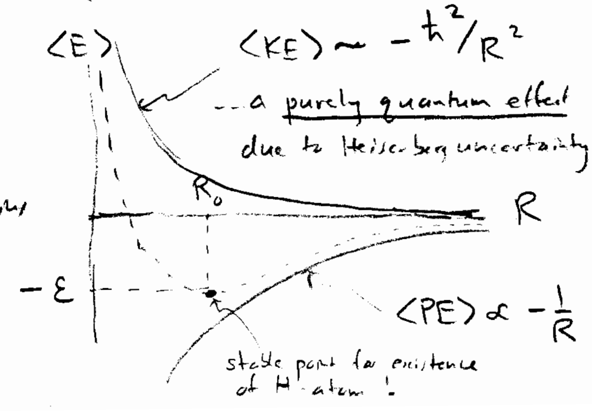
$\langle PE\rangle \sim -\dfrac{e^2}{4\pi\epsilon_0 R}$, where $\langle..\rangle$ denotes "time average"

$$\langle KE\rangle = \left\langle \frac{p^2}{2m} \right\rangle = \left\langle \frac{\Delta p^2}{2m} \right\rangle \qquad (\text{since } \langle \vec{p}\rangle = 0)$$

$$\gtrsim \frac{\hbar^2}{2mR^2}, \quad \text{using } \Delta p \gtrsim \frac{\hbar}{\Delta x} \text{ with } \Delta x \sim R$$

$$\therefore\ \langle E\rangle \gtrsim \underbrace{\frac{\hbar^2}{2mR^2} - \frac{e^2}{4\pi\epsilon_0 R}}$$

minimizing this function of R gives R_0 = Bohr radius and $\mathcal{E} = \dfrac{me^4}{8\epsilon_0^2 h^2}$ as above



(b)

plane (4) $\longrightarrow v_{43} = v$

bird (3) $\longrightarrow v_{32} = v$

train (2) $\longrightarrow v_{21} = v$

ground (1)

What is v_{41} (velocity of 4 relative to 1) ?

$$v_{42} = \frac{v_{43}+v_{32}}{1+v_{43}v_{32}/c^2} = \frac{v+v}{1+v^2/c^2} = \frac{2v}{1+v^2/c^2}$$

$$\therefore \; v_{41} = \frac{v_{42}+v_{21}}{1+v_{42}v_{21}/c^2} = \frac{\frac{2v}{(1+v^2/c^2)} + v}{1 + \frac{2v}{1+v^2/c^2}\frac{v}{c^2}} = \frac{2v + v(1+v^2/c^2)}{1+\frac{v^2}{c^2} + 2\frac{v^2}{c^2}}$$

$$= \frac{3v + v^3/c^2}{1+3v^2/c^2}$$

$$\boxed{\frac{v_{41}}{v} = \frac{3+v^2/c^2}{1+3v^2/c^2}}$$

Newtonian limit ($c \to \infty$): $\frac{v_{41}}{v} \approx 3$ ✓

Limit as $v \to c$: $\frac{v_{41}}{v} \to \frac{3+1}{1+3} = 1$, ie $v_{41} \to c$ ✓

Problem 7 : Commutators in Quantum Mechanics

Concept : $[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$

...a simple identity you should prove.

(a) Given: $[\hat{p},\hat{x}] = \frac{\hbar}{i}$

$$\therefore\ [\hat{p}^2,\hat{x}] = \hat{p}[\hat{p},\hat{x}] + [\hat{p},\hat{x}]\hat{p} = \frac{\hbar}{i}(2\hat{p})$$

$$[\hat{p}^3,\hat{x}] = \hat{p}[\hat{p}^2,\hat{x}] + [\hat{p},\hat{x}]\hat{p}^2 = \frac{\hbar}{i}(2\hat{p}^2+\hat{p}^2) = \frac{\hbar}{i}(3\hat{p}^2)$$

$$[\hat{p}^4,\hat{x}] = \hat{p}[\hat{p}^3,\hat{x}] + [\hat{p},\hat{x}]\hat{p}^3 = \frac{\hbar}{i}(3\hat{p}^3+\hat{p}^3) = \frac{\hbar}{i}(4\hat{p}^3)$$

$$\vdots$$

$$\therefore\ [\hat{p}^n,\hat{x}] = \frac{\hbar}{i}\, n\,\hat{p}^{n-1}$$

$$\therefore\ [e^{\frac{i}{\hbar}a\hat{p}},\hat{x}] = \left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{\hbar}a\hat{p}\right)^n,\hat{x}\right]$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{\hbar}a\right)^n[\hat{p}^n,\hat{x}]$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{\hbar}a\right)^n\frac{\hbar}{i}\,n\,\hat{p}^{n-1}$$

$n=0$ term killed by n

$$= a\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\left(\frac{i}{\hbar}a\hat{p}\right)^{n-1} = a\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{i}{\hbar}a\hat{p}\right)^m$$

$$= a\,e^{\frac{i}{\hbar}a\hat{p}}$$

alternate derivation:

Represent $[\hat{p},\hat{x}] = \frac{\hbar}{i}$ with $\hat{x} = -\frac{\hbar}{i}\frac{\partial}{\partial p}$, $\hat{p} = p$

Then: $[e^{\frac{i}{\hbar}a\hat{p}},\hat{x}]\psi(p) = e^{\frac{i}{\hbar}ap}\left(-\frac{\hbar}{i}\frac{\partial\psi}{\partial p}\right) + \frac{\hbar}{i}\frac{\partial}{\partial p}\left(e^{\frac{i}{\hbar}ap}\psi\right)$

$$= \frac{\hbar}{i}\frac{\partial}{\partial p}\left(e^{\frac{i}{\hbar}ap}\right)\psi = a\,e^{\frac{i}{\hbar}ap}\psi$$

$$\Rightarrow\ [e^{\frac{i}{\hbar}a\hat{p}},\hat{x}] = a\,e^{\frac{i}{\hbar}a\hat{p}}\quad\checkmark$$

(b) Given: $[\hat{A}, \hat{B}] = 1$

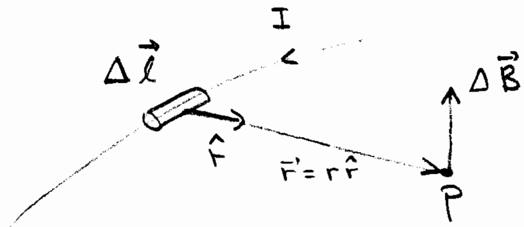
$$\therefore \; [\hat{A}, \hat{B}^2] = [\hat{A}, \hat{B}]\hat{B} + \hat{B}[\hat{A}, \hat{B}] \quad \leftarrow \text{similar to identity above}$$

$$= \hat{B} + \hat{B}$$

$$= 2\hat{B} \,.$$

Problem 8 : Magnetic Fields

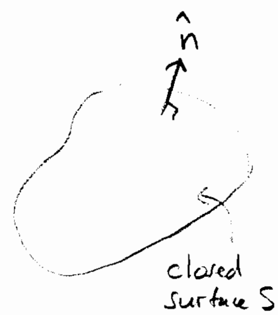
Concepts : (i) Biot-Savart Law



$$\Delta\vec{B} = \frac{\mu_0 I}{4\pi r^2}\,\Delta\vec{l}\times\hat{r}$$

(ii) Gauss' Law

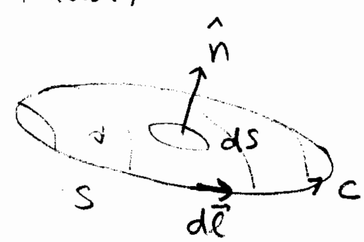
$$\vec{\nabla}\cdot\vec{B} = 0 \implies \oint_S \vec{B}\cdot\hat{n}\,dS = 0$$



(iii) Ampère's Law (for steady currents / fields)

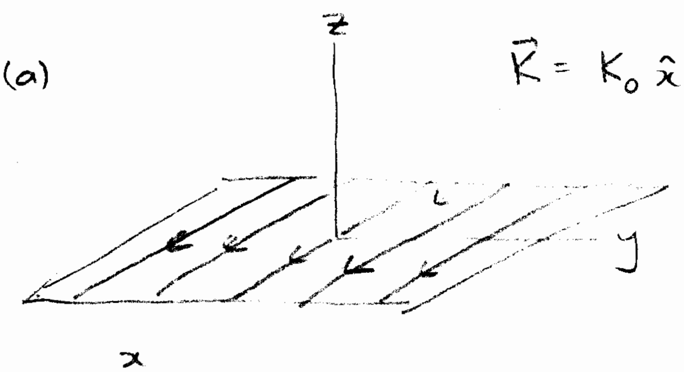
$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0\int_S \vec{J}\cdot\hat{n}\,dS$$

$$= \mu_0\,(\text{current enclosed})$$



The Problem

(a)



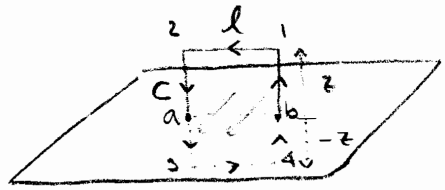
$\vec{K} = K_0\,\hat{x}$

This is a steady surface current (K_0 is Amps/meter) in free space.

- Biot Savart Law + Symmetry

$$\implies \vec{B} = \begin{cases} -\hat{y}\,f(z), & z > 0 \\ +\hat{y}\,f(-z), & z < 0 \end{cases}$$

where $f(z)$ is some function [actually, we know $f(z)$ = constant because, with a current sheet of infinite extent, there is no dimensionful parameter to set a "scale"]



$$\oint_C \vec{B}\cdot d\vec{l} = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1\right)(\vec{B}\cdot d\vec{l})$$

Integrand $\propto \hat{y}\cdot\hat{z} = 0$

$$\therefore \oint_C \vec{B}\cdot d\vec{l} = \int_1^2 [-\hat{y} f(z)]\cdot[\hat{y}\,dy] + \int_3^4 [+\hat{y} f(-(-z))]\cdot[\hat{y}\,dy]$$

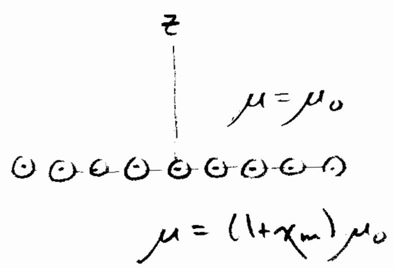
$$= + f(z)\,l + f(z)\,l = 2 f(z)\,l$$

$$\mu_0 \int_S \vec{J}\cdot\hat{n}\,dS = \mu_0 \int_a^b \vec{K}\cdot\hat{x}\,dy = \mu_0 K_0 l$$

$$\therefore 2 f(z)\,\cancel{l} = \mu_0 K_0 \cancel{l} \implies f(z) = \frac{\mu_0 K_0}{2} = \text{constant} \quad \checkmark$$

$$\implies \boxed{\vec{B} = \begin{cases} -\hat{y}\dfrac{\mu_0 K_0}{2}, & z > 0 \\ +\hat{y}\dfrac{\mu_0 K_0}{2}, & z < 0 \end{cases}}$$

(b)



We are given two facts:

(i) $\vec{\nabla}\times\vec{H} = \vec{J}_{free} \implies \oint_C \vec{H}\cdot d\vec{l} = \int_S \vec{J}_{free}\cdot\hat{n}\,dS$

In analogy with our previous results we have:

$$\vec{H} = \begin{cases} -\hat{y}\,H_+, & z > 0 \\ +\hat{y}\,H_-, & z < 0 \end{cases}$$

where $\oint_C \vec{H}\cdot d\vec{l} = H_+ l + H_- l = \int_S \vec{J}_{free}\cdot\hat{n}\,dS = K_0 l$

$\implies \boxed{H_+ + H_- = K_0}$, i.e. $H_y(z>0) - H_y(z<0) = -H_+ - H_- = -K_0$

--- $\vec{H}_{tan}$ is discontinuous by $-K_0$ above/below $z=0$

(ii) $\vec{B} = \begin{cases} -\hat{y}\,B_0 = -\hat{y}\,\mu_0 H_+ & , z>0 \\ +\hat{y}\,B_0 = +\hat{y}\,(1+\chi_m)\mu_0 H_- & , z<0 \end{cases}$

$\uparrow$ $\vec{B}(z) = -\vec{B}(-z)$ (SHII)

$\uparrow$ $\vec{B} = \mu\vec{H} = \begin{cases} \mu_0 \vec{H} & , z>0 \\ (1+\chi_m)\mu_0\vec{H} & , z<0 \end{cases}$

Thus we have:

$$\left.\begin{aligned} H_+ + H_- &= K_0 \\ H_+ &= \frac{B_0}{\mu_0} \\ H_- &= \frac{B_0}{(1+\chi_m)\mu_0} \end{aligned}\right\} \begin{array}{l} 3 \text{ eqns, } 3 \text{ unknowns} \\ (H_+, H_-, B_0) \end{array}$$

$$\therefore\ K_0 = H_+ + H_- = \frac{B_0}{\mu_0}\left(1 + \frac{1}{1+\chi_m}\right) = \frac{B_0}{\mu_0}\left(\frac{2+\chi_m}{1+\chi_m}\right)$$

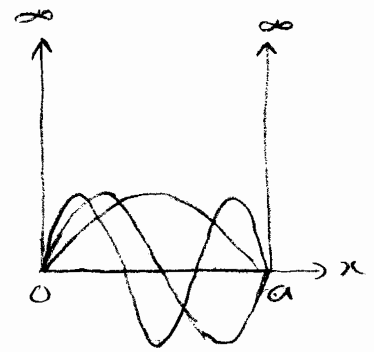
$$\Rightarrow\ \frac{B_0}{\mu_0} = \left(\frac{1+\chi_m}{2+\chi_m}\right) K_0$$

$$\Rightarrow\ H_+ = \left(\frac{1+\chi_m}{2+\chi_m}\right) K_0 \quad , \quad H_- = \frac{1}{2+\chi_m} K_0$$

$$\vec{B} = \begin{cases} -\hat{y}\left(\frac{1+\chi_m}{2+\chi_m}\right)\mu_0 K_0 \xrightarrow{\chi_m=0} -\hat{y}\,\frac{\mu_0 K_0}{2} & , z>0 \\ +\hat{y}\left(\frac{1+\chi_m}{2+\chi_m}\right)\mu_0 K_0 \xrightarrow{\chi_m=0} +\hat{y}\,\frac{\mu_0 K_0}{2} & , z<0 \end{cases}$$

$$\vec{H} = \begin{cases} -\hat{y}\left(\frac{1+\chi_m}{2+\chi_m}\right) K_0 \xrightarrow{\chi_m=0} -\hat{y}\,\frac{K_0}{2} & , z>0 \\ +\hat{y}\left(\frac{1}{2+\chi_m}\right) K_0 \xrightarrow{\chi_m=0} +\hat{y}\,\frac{K_0}{2} & , z<0 . \end{cases}$$

Problem 9 : Quantum Dynamics in a Square Well



$$\left.\begin{aligned} \phi_n(x) &= \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \\ E_n &= \frac{n^2\pi^2\hbar^2}{2ma^2} \end{aligned}\right\} n = 1, 2, 3, \ldots$$

(a) General sol'n: $\psi(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\frac{i}{\hbar}E_n t}$

where $\sum_{n=1}^{\infty} |a_n|^2 = 1$

Given: $|a_1|^2 = |a_2|^2 \neq 0$ (all other $a_n = 0$)

$\Rightarrow |a_1| = |a_2| = \frac{1}{\sqrt{2}}$

choose: $a_1 = \frac{1}{\sqrt{2}}$, $a_2 = \frac{1}{\sqrt{2}} e^{i\theta}$ (only relative phase, θ, can matter).

$$\therefore \boxed{\psi(x,t) = \frac{1}{\sqrt{2}}\left[\phi_1(x) e^{-\frac{i}{\hbar}E_1 t} + \phi_2(x) e^{-\frac{i}{\hbar}E_2 t + i\theta} \right]}$$

(b) $\langle x \rangle \equiv \int_0^a dx\, \psi^*(x,t)\, x\, \psi(x,t)$

$$= \int_0^a dx\, \frac{1}{2} x \left[\phi_1 e^{i\omega_1 t} + \phi_2 e^{i(\omega_2 t - \theta)} \right]\left[\phi_1 e^{-i\omega_1 t} + \phi_2 e^{-i(\omega_2 t - \theta)} \right]$$

where $\omega_n \equiv \frac{E_n}{\hbar}$.

$$= \frac{1}{2}\int_0^a dx\, x \left[\phi_1^2 + \phi_2^2 + \phi_1\phi_2 \left(e^{i[(\omega_1-\omega_2)t+\theta]} + e^{-i[(\omega_1-\omega_2)t+\theta]} \right) \right]$$

$$= \frac{1}{2}\int_0^a dx\, x \left[\phi_1^2 + \phi_2^2 + 2\phi_1\phi_2 \underbrace{\cos[(\omega_2-\omega_1)t - \theta]}_{= f(t)} \right]$$

$$\langle x \rangle = \frac{1}{a} \int_0^a dx\, x \left[\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right) + 2f \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) \right]$$

$$\alpha \equiv \frac{\pi x}{a} \Rightarrow d\alpha = \frac{\pi}{a} dx \;,\; \int_{x=0}^{a} = \int_{\alpha=0}^{\pi}$$

$$= \frac{a}{\pi^2} \int_0^\pi d\alpha\, \alpha \left[\sin^2\alpha + \sin^2 2\alpha + 2f \sin\alpha \sin 2\alpha \right]$$

<u>use:</u> $\sin^2\alpha = \dfrac{1-\cos 2\alpha}{2}$, $\sin^2 2\alpha = \dfrac{1-\cos 4\alpha}{2}$

$$\sin\alpha \sin\beta = \frac{e^{i\alpha} - e^{-i\alpha}}{2i} \times \frac{e^{i\beta} - e^{-i\beta}}{2i}$$

$$= -\frac{1}{4} \left[e^{i(\alpha+\beta)} + e^{-i(\alpha+\beta)} - e^{i(\alpha-\beta)} - e^{-i(\alpha-\beta)} \right]$$

$$= -\frac{1}{2} \left[\cos(\alpha+\beta) - \cos(\alpha-\beta) \right]$$

$$\therefore \sin\alpha \sin 2\alpha = -\frac{1}{2} \left[\cos 3\alpha - \cos\alpha \right]$$

$$\langle x \rangle = \frac{a}{2\pi^2} \int_0^\pi d\alpha\, \alpha \left[(1-\cos 2\alpha) + (1-\cos 4\alpha) - 2f(\cos 3\alpha - \cos\alpha) \right]$$

$$= \frac{a}{2\pi^2} \int_0^\pi d\alpha\, \alpha \left\{ (2 - \cos 2\alpha - \cos 4\alpha) + 2f(\cos\alpha - \cos 3\alpha) \right\}$$

<u>use:</u> $\dfrac{d}{dz}(z \sin z + \cos z) = z\cos z + \cancel{\sin z} - \cancel{\sin z} = z \cos z$

$$\Rightarrow I_n \equiv \int_0^\pi d\alpha\, \alpha \cos n\alpha$$

let $z = n\alpha$, $dz = n\, d\alpha$, $\displaystyle\int_{\alpha=0}^{\pi} = \int_{z=0}^{n\pi}$

$$I_n = \int_0^{n\pi} \frac{dz}{n} \frac{z}{n} \cos z = \frac{1}{n^2} \int_0^{n\pi} dz\, z \cos z$$

$$= \frac{1}{n^2} \int_0^{n\pi} d(z \sin z + \cos z) = \frac{1}{n^2} (z \sin z + \cos z)\Big|_0^{n\pi}$$

$$= \frac{1}{n^2} \left(n\pi \sin n\pi + \cos n\pi - 0 - 1 \right) = \frac{\cos n\pi - 1}{n^2}$$

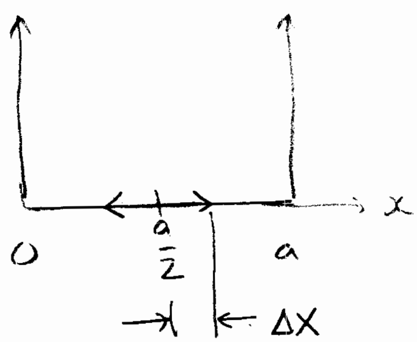
$$\Rightarrow I_1 = -2 \;,\; I_2 = 0 \;,\; I_3 = -\frac{2}{9} \;,\; I_4 = 0$$

$$\langle x \rangle = \frac{a}{2\pi^2} \left\{ \alpha^2 \Big|_0^\pi - \cancelto{0}{I_2} - \cancelto{0}{I_4} + 2f\,(I_1 - I_3) \right\}$$

$$= \frac{a}{2\pi^2} \left\{ \pi^2 + 2f\left(-2 + \frac{2}{9}\right) \right\}$$

$$= \frac{a}{2} \left\{ 1 - \frac{32}{9\pi^2} f(t) \right\}$$

$$\boxed{\begin{array}{l} \langle x \rangle = \dfrac{a}{2} \left\{ 1 - \dfrac{32}{9\pi^2} \cos\left[(\omega_2 - \omega_1)t - \theta \right] \right\} \\ \text{where } \omega_2 - \omega_1 = \dfrac{1}{\hbar}(E_2 - E_1) = \dfrac{3\pi^2 \hbar}{2ma^2} \end{array}}$$



$\langle x \rangle$ oscillates about $x = \frac{a}{2}$ with frequency $\Delta\omega \equiv \omega_2 - \omega_1$ and amplitude $\Delta X = \left(\frac{32}{9\pi^2}\right)\frac{a}{2} \approx (0.36)\frac{a}{2}$

NB: this is a <u>non-stationary</u> state ; if the particle were charged, it is precisely in such a non-stationary superposition state that it would be emitting or absorbing a photon of frequency $\Delta\omega$ in the process of making a transition between the ground and first excited states. <u>Semiclassically</u>, the charge distribution "sloshes back and forth" in the process of emitting or absorbing electromagnetic radiation.

(c) At $t = 0$: $\langle x \rangle = \frac{a}{2}\left\{ 1 - \frac{32}{9\pi^2}\cos\theta \right\}$

$$= \frac{a}{2}\left\{ 1 + \frac{32}{9\pi^2} \right\} \quad \text{for } \theta = \pi$$

$\Rightarrow \psi(x,t) =$ as given in part (a), with $\theta = \pi$

(d) $$\langle p \rangle = \int_0^a dx\, \psi^*(x,t)\, \frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x,t)$$

$$= \frac{1}{2}\frac{\hbar}{i}\int_0^a dx \left[\phi_1 e^{i\omega_1 t} + \phi_2 e^{i(\omega_2 t-\theta)}\right]\frac{\partial}{\partial x}\left[\phi_1 e^{-i\omega_1 t} + \phi_2 e^{-i(\omega_2 t-\theta)}\right]$$

$$= \frac{\hbar}{2i}\int_0^a dx \left\{ \overset{①}{\phi_1 \frac{d\phi_1}{dx}} + \overset{②}{\phi_2\frac{d\phi_2}{dx}} + \overset{③}{\phi_1\frac{d\phi_2}{dx}} e^{i[(\omega_1-\omega_2)t+\theta]} + \underset{④}{\phi_2\frac{d\phi_1}{dx}} e^{-i[(\omega_1-\omega_2)t+\theta]} \right\}$$

$$① + ② = \frac{\hbar}{2i}\int_0^a dx \left\{ \frac{1}{2}\frac{d\phi_1^2}{dx} + \frac{1}{2}\frac{d\phi_2^2}{dx} \right\}$$

$$= \frac{\hbar}{4i}\left\{\phi_1^2 + \phi_2^2\right\}\Big|_0^a$$

$$= \frac{\hbar}{4i}\frac{2}{a}\left\{\sin^2\frac{\pi x}{a} + \sin^2\frac{2\pi x}{a}\right\}\Big|_0^a$$

$$= 0$$

$$③ = \frac{\hbar}{2i}\frac{2}{a}\int_0^a dx\, \sin\left(\frac{\pi x}{a}\right)\frac{d}{dx}\sin\left(\frac{2\pi x}{a}\right) e^{+i[\ ]}$$

$$= \frac{\hbar}{2i}\frac{2}{a} e^{+i[\ ]}\frac{2\pi}{a}\int_0^a dx\, \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi x}{a}\right)$$

use: $$\sin\alpha\cos\beta = \frac{e^{i\alpha}-e^{-i\alpha}}{2i} \times \frac{e^{i\beta}+e^{-i\beta}}{2}$$

$$= \frac{1}{4i}\left[e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)} + e^{i(\alpha-\beta)} - e^{-i(\alpha-\beta)}\right]$$

$$= \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$= \frac{\hbar}{2i}\frac{2}{a} e^{+i[\ ]}\frac{2\pi}{a}\frac{1}{2}\int_0^a dx \left[\sin\frac{3\pi x}{a} - \sin\frac{\pi x}{a}\right]$$

$$= \frac{\hbar}{2i}\frac{2\pi}{a^2} e^{+i[\ ]}\left\{-\frac{a}{3\pi}\cos\frac{3\pi x}{a} + \frac{a}{\pi}\cos\frac{\pi x}{a}\right\}_0^a$$

$$= \frac{\hbar}{2i}\frac{2}{a} e^{+i[\ ]}\left\{-\frac{1}{3}(\cos 3\pi - 1) + (\cos\pi - 1)\right\}$$

$$= -\frac{\hbar}{i}\left(\frac{4}{3a}\right)e^{+i[\ ]}$$

$$④ = \frac{\hbar}{2i}\,\frac{2}{a}\int_0^a dx\, \sin\left(\frac{2\pi x}{a}\right)\frac{d}{dx}\sin\left(\frac{\pi x}{a}\right) e^{-i[\,]}$$

$$= \frac{\hbar}{2i}\,\frac{2}{a}\,e^{-i[\,]}\,\frac{\pi}{a}\int_0^a dx\, \sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right)$$

$$= \frac{\hbar}{i}\,\frac{\pi}{a^2}\,e^{-i[\,]}\,\frac{1}{2}\int_0^a dx\left[\sin\left(\frac{3\pi x}{a}\right)+\sin\left(\frac{\pi x}{a}\right)\right]$$

$$= \frac{\hbar}{i}\,\frac{\pi}{2a^2}\,e^{-i[\,]}\left\{-\frac{a}{3\pi}\cos\left(\frac{3\pi x}{a}\right)-\frac{a}{\pi}\cos\left(\frac{\pi x}{a}\right)\right\}_0^a$$

$$= \frac{\hbar}{i}\,\frac{1}{2a}\,e^{-i[\,]}\left\{-\frac{1}{3}\left(\cos 3\pi - 1\right)-\left(\cos\pi - 1\right)\right\}$$

$$= \frac{\hbar}{i}\left(\frac{4}{3a}\right)e^{-i[\,]}$$

$$\therefore \langle p\rangle = -\frac{\hbar}{i}\,\frac{4}{3a}\left(e^{+i[\,]}-e^{-i[\,]}\right)$$

$$= +\hbar\,\frac{8}{3a}\sin\left[(\omega_2-\omega_1)t-\theta\right]$$

$$\boxed{\begin{aligned}\langle p\rangle &= \frac{8\hbar}{3a}\sin(\Delta\omega t-\theta)\\ &= -\frac{8\hbar}{3a}\sin(\Delta\omega t)\quad \text{for}\quad \theta=\pi\end{aligned}}$$

check: $$m\frac{d\langle x\rangle}{dt} = m\,\frac{a}{2}\,\frac{32}{9\pi^2}\,\Delta\omega\,\sin(\Delta\omega t-\theta)$$

$$= \cancel{m}\,\frac{\cancel{a}}{2}\,\frac{32}{9\cancel{\pi^2}}\,\frac{3\cancel{\pi^2}\hbar}{2\cancel{m}\cancel{a^2}}\sin(\Delta\omega t-\theta)$$

$$= \frac{8\hbar}{3a}\sin(\Delta\omega t-\theta)$$

$$= \langle p\rangle$$ ✓ (Ehrenfest's Theorem)

Problem 10 : Fourier Analysis

Concept :
$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, e^{-i\omega t} f(t)$$
$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, e^{+i\omega t} F(\omega)$$
i.e. $$\delta(\omega-\omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i(\omega-\omega')t}$$

The Problem

(a) $$f(t) = \begin{cases} 0 & , t<0 \\ A e^{-at}\cos\omega_1 t & , t>0 \end{cases}$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} dt\, e^{-i\omega t} A e^{-at} \cos\omega_1 t$$
$$= \frac{1}{\sqrt{2\pi}} \frac{A}{2} \int_0^{\infty} dt \left(e^{-[a+i(\omega-\omega_1)]t} + e^{-[a+i(\omega+\omega_1)]t} \right)$$
$$= \frac{1}{\sqrt{2\pi}} \frac{A}{2} \left\{ \frac{e^{-[a+i(\omega-\omega_1)]t}}{-[a+i(\omega-\omega_1)]} + \frac{e^{-[a+i(\omega+\omega_1)]t}}{-[a+i(\omega+\omega_1)]} \right\}_0^{\infty}$$
$$= \frac{1}{\sqrt{2\pi}} \frac{A}{2} \left\{ \frac{1}{a+i(\omega-\omega_1)} + \frac{1}{a+i(\omega+\omega_1)} \right\}$$

(b) $a \ll \omega_1$

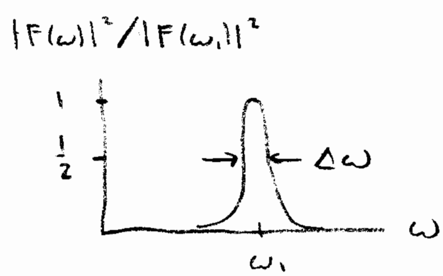
Note: $|a+i(\omega+\omega_1)| = \sqrt{a^2+(\omega+\omega_1)^2} > \omega_1$ for all $\omega > 0$

Compare with :

$|a+i(\omega-\omega_1)| = \sqrt{a^2+(\omega-\omega_1)^2}$ ... can be as small as "a" (for $\omega=\omega_1$)

$\therefore$ For $\omega > 0$: $F(\omega) \approx \frac{1}{\sqrt{2\pi}} \frac{A}{2} \frac{1}{a+i(\omega-\omega_1)}$

$$|F(\omega)|^2 \approx \frac{A^2}{8\pi} \frac{1}{a^2+(\omega-\omega_1)^2}$$
$$|F(\omega_1)|^2 \approx \frac{A^2}{8\pi}\frac{1}{a^2}$$



F.W.H.M. (Full Width at Half Maximum):

$$\frac{|F(\omega_\pm)|^2}{|F(\omega_1)|^2} = \frac{1}{2} \quad ; \text{ solve for } \omega_\pm :$$

$$\frac{a^2}{a^2+(\omega_\pm-\omega_1)^2} = \frac{1}{2} \Rightarrow \omega_\pm = \omega_1 \pm a \Rightarrow \boxed{\Delta\omega = 2a}$$

(c) $$f(t) = \begin{cases} 0 , & t<0 \\ Ae^{-at}\cos\omega_1 t + Ae^{-at}\cos\omega_2 t , & t>0 \end{cases}$$

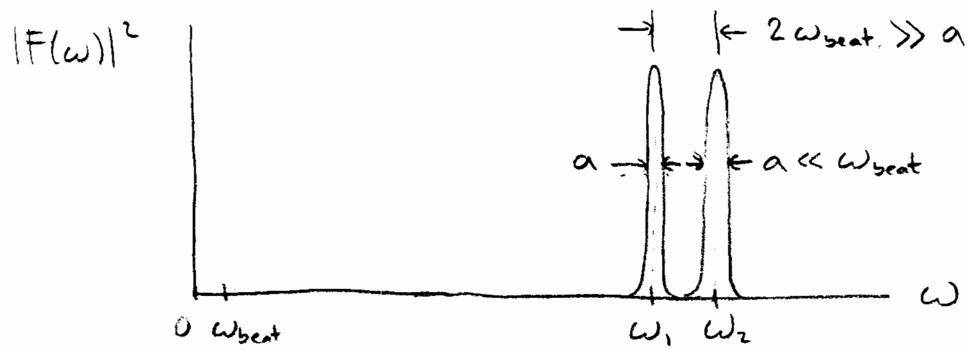
$$\cos\alpha\cos\beta = \frac{e^{i\alpha}+e^{-i\alpha}}{2}\cdot\frac{e^{i\beta}+e^{-i\beta}}{2} = \frac{1}{4}\left(e^{i(\alpha+\beta)}\right.$$
$$= \frac{1}{4}\left(e^{i(\alpha+\beta)}+e^{-i(\alpha+\beta)}+e^{i(\alpha-\beta)}+e^{-i(\alpha-\beta)}\right)$$
$$= \frac{1}{2}\left\{\cos(\alpha+\beta)+\cos(\alpha-\beta)\right\}$$

$$\therefore f(t) = 2Ae^{-at}\cos\left[\tfrac{1}{2}(\omega_2-\omega_1)t\right]\cos\left[\tfrac{1}{2}(\omega_1+\omega_2)t\right] , \quad t>0$$

$$\omega_2 \approx \omega_1 , \quad \omega_{beat} \equiv \frac{1}{2}(\omega_2-\omega_1)$$

$$\boxed{\begin{array}{l} f(t) \approx 2Ae^{-at}\cos\omega_{beat}t\,\cos\omega_1 t , \quad t>0 \\ \text{where } a \ll \omega_{beat} \ll \omega_1 \end{array}}$$

(d) $$F(\omega) \approx \frac{1}{\sqrt{2\pi}}\frac{A}{2}\left\{\frac{1}{a+i(\omega-\omega_1)}+\frac{1}{a+i(\omega-\omega_2)}\right\}$$



(e) There is no peak at $\omega = \omega_{beat}$. To understand this, consider:

$$[f(t)]^2 \approx [4\cos^2\omega_{beat}t][Ae^{-at}\cos\omega_1 t]^2$$

time average of "beat envelope" (

) is 2

i.e. the average power is twice that of a single $Ae^{-at}\cos\omega_1 t$ signal

-- sometimes the ω_1 & ω_2 signals are in phase: constructive interference, amplitude is double, power is quadruple

-- sometimes the ω_1 & ω_2 signals are out of phase: destructive interference, amplitude & power are zero.

-- average of quadruple & zero is two.

i.e. There is no power in the signal at $\omega = \omega_{beat}$; when we hear beats we are not hearing a sine wave at $\omega = \omega_{beat}$; we are hearing a sum of sine waves (at ω_1 & ω_2), which slowly alternate between being in phase (4 x the power of a single sine wave) and out of phase (0 power).