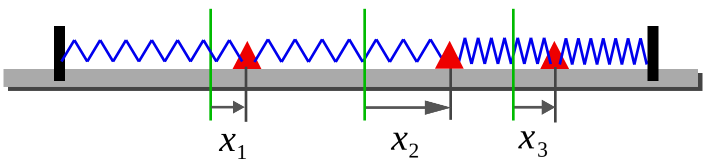
Coupled Oscillations

Definition:

• linear chain of n identical bodies (mass m) connected to one another and to fixed endpoints by identical ideal springs (spring constant k)

• distances from equilibrium x_i, i=1...n

• zero initial velocities; friction ignored



Prerequisites:

• the [simple harmonic oscillator](#)

• [Newtonian mechanics](#) in one dimension

Why study it?

• an excellent illustration of the physical significance of eigenvalues and eigenvectors

• a prerequisite to studying waves in continuous media

Summary:

The positions as functions of time are

$$\boxed{\vec{x}(t) = \sum_{i=1}^{n} c_i \vec{v}_i \cos \omega_i t} \quad ,$$

where for each i=1...n:

• the vector $\vec{v}_i$ represents the *normal mode* with

• frequency ω_i and

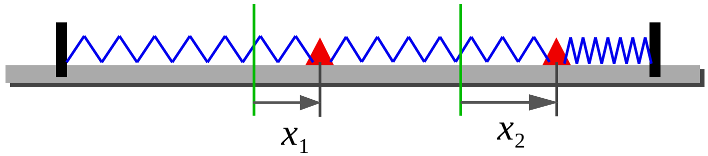
• amplitude c_i .

In words: the general motion is the sum of simpler motions (the normal modes), each of which has a definite frequency. If the system starts out in a particular normal mode, it remains there.

[Go to derivation](#)

[Go to Java™ applet](#)

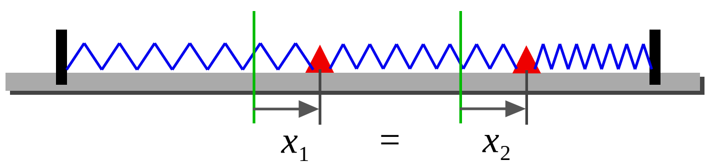
Here's an interesting situation. We have two bodies on our frictionless track, joined up with ideal springs like this:



The masses are identical, as are the springs. We measure the two distances from the equilibrium positions as shown.

Let's think about the answer before we use mathematics. In general, the motion will be rather complicated. But there are two kinds of motion which distinguish themselves by being very simple. These are very important, so we'll think about them first.

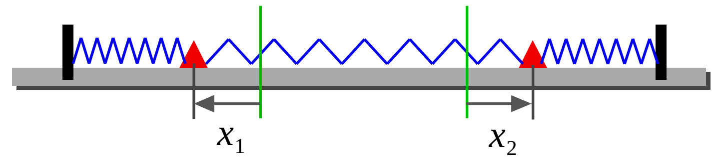
In one kind of simple motion, x_1 and x_2 remain equal. The whole thing oscillates "back and forth", and the middle spring is never stretched:



We can easily tell what the frequency of oscillation will be, because from the point of view of either mass, it's as if the middle spring were not present. Each mass is effectively attached to a single spring with constant k, so the frequency is

$$\omega_0 = \sqrt{\frac{k}{m}} \ .$$

In the other kind of simple motion, x_1 and x_2 remain exactly opposite. The motion is of the "in and out" type:



It is again relatively easy to guess the frequency. Look at one of the bodies in the above figure. It is attached to one spring compressed by amount x_1, and to another which is stretched by twice that amount. The net restoring force on the body is therefore *three* times as large as in the previous case, so it's as if k were replaced by $3k$. The frequency will be

$$\omega = \sqrt{\frac{3k}{m}} = \sqrt{3}\,\omega_0 \ .$$

These simple motions are called *normal modes* of the system. They have the property that if the system starts out in one of the normal modes, then it will remain in that mode. The resources for this section contain movies of the lower- and higher-frequency normal modes, as well as a movie of a more complicated motion of the system.

Now let's see how the application of Newton's law proceeds. What is the force on body 1? Well, it has two parts, $-kx_1$ due to the leftmost spring and $+k(x_2-x_1)$ due to the center spring. To see this last point, imagine that $x_2>x_1$; in that case, the center spring is stretched and it pulls body 1 in the positive direction. The net force on body 1 is therefore $-2kx_1+kx_2$.

We obtain the net force on body 2 by a similar analysis. Hence, we arrive at the equations of motion:

$$m\ddot{x}_1 = -2kx_1+kx_2$$

$$m\ddot{x}_2 = +kx_1-2kx_2$$

This is a pair of *coupled* differential equations. In cases like this, the strategy is to try and find linear combinations of x_1 and x_2 which obey uncoupled equations. This turns out to be very easy in our example: adding and subtracting the two equations gives

$$m(\ddot{x}_1+\ddot{x}_2) = -k(x_1+x_2)$$

$$m(\ddot{x}_1-\ddot{x}_2) = -3k(x_1-x_2)$$

We immediately see that, however complicated x_1 and x_2 may be individually, their sum x_1+x_2 always oscillates with frequency ω_0 and their difference x_1-x_2 always oscillates with frequency $\sqrt{3}\omega_0$.

This agrees with what we deduced above. For the "back and forth" mode, x_1+x_2 is nonzero and oscillates with frequency ω_0. For the "in and out" mode, x_1-x_2 is nonzero and oscillates at the higher frequency.

From our earlier section on the harmonic oscillator, the solutions may be written

$$x_1+x_2 = A_1\cos(\omega_0 t-\delta_1)$$

$$x_1-x_2 = A_2\cos(\sqrt{3}\omega_0 t-\delta_2)$$

where the A's and δ's are determined by initial conditions. Solving for x_1, for example, we find

$$x_1 = \frac{1}{2}\left(A_1\cos(\omega_0 t-\delta_1)+A_2\cos(\sqrt{3}\omega_0 t-\delta_2)\right).$$

This shows that x_1 is a *superposition* of motions with two different frequencies. You can easily convince yourself that $x_1(t)$ is, in general, *not periodic*. This is because the ratio of the two frequencies is $\sqrt{3}$, which is not a rational number. That is, it is not of the form n/m where n and m are integers. So there's no way that the motion can repeat itself after any number of periods.

Of course, the motion is periodic if either of the A's is zero - the system is then in one of its normal modes!

Eigenvalues and eigenvectors

Let's look a little more closely at our pair of coupled equations. We are going to use another language to derive the characteristics of the solution: the language of matrices, eigenvalues and eigenvectors.

Let's write our equations in matrix form:

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} = -\frac{k}{m}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} .$$

Let's symbolize the 2×2 matrix in this equation by M. What are its *eigenvalues*? The eigenvalues are the solutions λ (read "lambda") of the *characteristic equation*

$$\det\left(\lambda\mathbb{I}-M\right)=0 ,$$

where "det" means *determinant* and $\mathbb{I}$ stands for the 2×2 unit matrix:

$$\mathbb{I} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} .$$

We have

$$0 = \det\begin{bmatrix}\lambda-2 & 1\\ 1 & \lambda-2\end{bmatrix} = (\lambda-2)^2-1 ,$$

which has two solutions,

$$\lambda=3 \quad \text{or} \quad \lambda=1 .$$

The frequencies of the normal modes are the *square roots* of these numbers, times ω_0. This is the recipe for finding the frequencies.

How do we find out what kind of motion the normal modes represent? We have to find the *eigenvectors* of the matrix *M*.

Let's find an eigenvector corresponding to the eigenvalue 3. This is a vector which, when multiplied by *M*, gives 3 times itself:

$$\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = 3\begin{bmatrix}a\\ b\end{bmatrix} .$$

When simplified, both the top and bottom components of this equation give the relation

$$a + b = 0 .$$

Hence,

$$\begin{bmatrix}a\\ b\end{bmatrix} \propto \begin{bmatrix}1\\ -1\end{bmatrix} .$$

(The eigenvector is only determined up to an overall multiplicative factor.)

The fact that the components of the eigenvector are *opposite* tells us that x_1 and x_2 are opposite in this mode, the one with frequency $\sqrt{3}\omega_0$.

An eigenvector with eigenvalue 1 satisfies

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix},$$

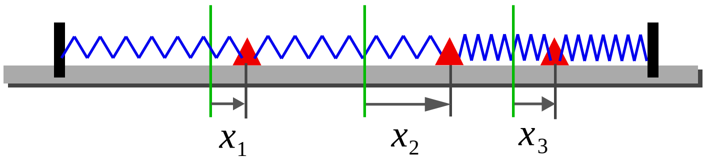
which has as its solution

$$\begin{bmatrix} a \\ b \end{bmatrix} \propto \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The fact that the components are *equal* tells us that x_1 and x_2 are equal in this mode, the one with frequency ω_0.

A case with three bodies

The reason we went through the above analysis with eigenvalues and eigenvectors is so that we can deal efficiently with more complicated cases, for example one in which there are *three* bodies. There, we will really begin to get a feeling for the physical significance of eigenvalues and eigenvectors.



It would be an excellent exercise for you to show that the equations of motion are

$$\begin{aligned} \ddot{x}_1 &= -\frac{k}{m}\left(2x_1 - x_2\right) \\ \ddot{x}_2 &= -\frac{k}{m}\left(-x_1 + 2x_2 - x_3\right) \\ \ddot{x}_3 &= -\frac{k}{m}\left(-x_2 + 2x_3\right). \end{aligned}$$

We must therefore find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

The characteristic equation is

$$\begin{aligned} \det\begin{bmatrix} \lambda-2 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-2 \end{bmatrix} &= (\lambda-2)\left((\lambda-2)^2-1\right)-(\lambda-2) \\ &= (\lambda-2)\left((\lambda-2)^2-2\right) \\ &= 0 \end{aligned}$$

which has three solutions:

$$\lambda = 2-\sqrt{2},\ \ 2, \text{ and }\ 2+\sqrt{2}\ \ .$$

The frequencies of the normal modes are the square roots of these numbers, times ω_0.

The eigenvectors are easily shown to be

$$\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} -1 \\ \sqrt{2} \\ -1 \end{bmatrix} .$$

The solution with eigenvalue 2 is easy to see; the middle body just sits there and the first and last bodies oscillate opposite to one another. Each has effectively two springs attached to it, hence the eigenvalue 2.

The other two solutions are more complicated, and you probably could not have guessed them before attempting the mathematics!

The resources for this section contain movies of the lowest-, middle-, and highest-frequency normal modes. And there is a Java™ applet which lets you set the number of bodies and their initial positions interactively.

Solution in general case with *n* bodies:

The matrix is an $n \times n$ matrix with 2's on the diagonal, -1's above and below it, and zeros elsewhere. It has n eigenvalues λ_i and n eigenvectors $\vec{v}_i$. The frequencies of the normal modes are

$$\omega_i = \sqrt{\lambda_i}\,\omega_0 \ ,$$

and the solution may be written as

$$\boxed{\vec{x}(t) = \sum_{i=1}^{n} c_i \vec{v}_i \cos\omega_i t} \quad ,$$

where we have taken the initial velocities of the bodies to be zero, for clarity.

In words, the above equation says that the motion of the chain, which is not necessarily periodic, is the sum of simpler motions, each of which has a definite frequency. These simpler motions are called the *normal modes*, and are represented by the vectors $\vec{v}_i$. The numbers c_i are coefficients which tell how much of a given normal mode is contained within the full motion. For this reason, they are called *weights* or *amplitudes*.

The eigenvectors have the important property that they are *orthogonal* to one another. This means that all dot products between eigenvectors are zero:

$$\vec{v}_i \cdot \vec{v}_j = 0 \ ,$$

except when i=j, of course. The dot product of a vector with itself is the square of the length of the vector, and it is convenient to adjust the vectors (*normalize* them) so that their lengths are 1:

$$\vec{v}_i \cdot \vec{v}_i = 1 \ .$$

Using these last two properties of the eigenvectors, we can calculate the coefficients c_i in terms of the initial values of the positions. Substituting t=0 in the equation for the positions, we get

$$\vec{x}(0) = \sum_{i=1}^{n} c_i \vec{v}_i \ .$$

Then, we take the dot product of both sides with a particular eigenvector, let's say the j'th one:

$$\vec{v}_j \cdot \vec{x}(0) = \sum_{i=1}^{n} c_i \left(\vec{v}_j \cdot \vec{v}_i\right) \ .$$

All terms in the sum are zero except when i=j, so we find using the normalization of the eigen-vectors

$$\boxed{\vec{v}_j \cdot \vec{x}(0) = c_j} \ .$$

This equation is valid for all j.

If the system is started out in one of its normal modes, let's say the k'th one, then $c_k \neq 0$ and all other c's are zero. The system remains in this normal mode at all later times:

$$\vec{x}(t) = c_k \vec{v}_k \cos\omega_k t \ .$$

Exercise:

Suppose the initial velocities of the bodies are nonzero. Show that the positions are given as functions of time by

$$\boxed{\vec{x}(t) = \sum_{i=1}^{n} \left(c_i \cos\omega_i t + d_i \sin\omega_i t\right) \vec{v}_i}$$

where

$$\boxed{c_i = \vec{x}(0) \cdot \vec{v}_i \quad \text{and} \quad d_i = \frac{\vec{v}(0) \cdot \vec{v}_i}{\omega_i}} \ .$$

Here, the initial velocities of the bodies are $\vec{v}(0)$.