Motion in a uniform electromagnetic field

Suppose a particle has mass m, electric charge q, and velocity $\vec{v}$, and moves with speed much less than the speed of light in a region containing electric and magnetic fields $\vec{E}$ and $\vec{B}$, respectively. Then its equation of motion is

$$\boxed{m\frac{d\vec{v}}{dt} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)} \quad .$$

The right-hand side of the above equation is called the ***Lorentz force***. The force due to the electric field is parallel to the electric field, and that due to the magnetic field is perpendicular to both the magnetic field and the velocity.

We will solve for the motion in fields which are the same everywhere (uniform), and do not change in time (constant). The equation of motion contains the velocity and its first derivative, so we will be able to solve for the velocity in terms of its initial value $\vec{v}_0$.

Here is a quick link to the result.

We will use an interesting method which uses matrices. It is not the only way to solve the equation of motion, but it is one of the most elegant.

It begins with the observation that the cross product $\vec{u}\times\vec{B}$, where $\vec{u}$ is any vector, can always be written as a matrix $\mathbf{B}$ multiplying the (column) vector $\vec{u}$ from the left:

$$\vec{u}\times\vec{B} = \mathbf{B}\vec{u} \quad ,$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{bmatrix} \quad .$$

If you don't believe this, just write out both sides of the above equation, remembering to write the vectors as column vectors.

So the equation of motion can be written

$$\frac{d\vec{v}}{dt} = \vec{E} + \mathbf{B}\vec{v} \quad ,$$

where we have temporarily absorbed a factor q/m into the fields. Differentiating once to get the acceleration, we obtain

$$\frac{d\vec{a}}{dt} = \mathbf{B}\vec{a} \quad .$$

Because the fields are constant and uniform, this can be solved immediately using matrix exponentials:

$$\vec{a} = \exp(\mathbf{B}t)\vec{a}_0 \ ,$$

where $\vec{a}_0 = \vec{E} + \mathbf{B}\vec{v}_0$ is the initial acceleration.

The exponential is defined by its power series:

$$\exp(\mathbf{B}t) = \mathbb{I} + \mathbf{B}t + \frac{1}{2!}(\mathbf{B}t)^2 + \frac{1}{3!}(\mathbf{B}t)^3 + \dots \ .$$

The matrix $\mathbf{B}$ has an interesting property that allow us to sum up the series in closed form: the cube of $\mathbf{B}$ is just given by

$$\mathbf{B}^3 = -B^2\mathbf{B} \ .$$

(Note that the scalar $B = \sqrt{\vec{B}\cdot\vec{B}}$.) This means that the terms with odd powers of $\mathbf{B}$are

$$\mathbf{B}t - \frac{1}{3!}B^2\mathbf{B}t^3 + \frac{1}{5!}B^4\mathbf{B}t^5 + \dots$$

$$= \frac{\mathbf{B}}{B}\left(Bt - \frac{1}{3!}B^3t^3 + \frac{1}{5!}B^5t^5 + \dots\right)$$

$$= \frac{\mathbf{B}}{B}\sin Bt \ .$$

The terms with even powers of $\mathbf{B}$ are

$$\mathbb{I} + \frac{1}{2!}\mathbf{B}^2t^2 - \frac{1}{4!}B^2\mathbf{B}^2t^4 + \dots$$

$$= \mathbb{I} + \frac{\mathbf{B}^2}{B^2}\left(\frac{1}{2!}B^2t^2 - \frac{1}{4!}B^4t^4 + \dots\right)$$

$$= \mathbb{I} + \frac{\mathbf{B}^2}{B^2}(1 - \cos Bt) \ .$$

Hence, everything sums up into the closed form

$$\vec{a} = \left(\mathbb{I} + \frac{1-\cos Bt}{B^2}\mathbf{B}^2 + \frac{\sin Bt}{B}\mathbf{B}\right)\left(\vec{E} + \mathbf{B}\vec{v}_0\right) \ .$$

If you multiply it out, you will find that the square of $\mathbf{B}$, acting on any vector $\vec{u}$, gives

$$\mathbf{B}^2\vec{u} = (\vec{B}\cdot\vec{u})\vec{B} - B^2\vec{u} = -B^2\vec{u}_\perp \ ,$$

where

$$\vec{u}_\perp = \vec{u} - \frac{\vec{u}\cdot\vec{B}}{B^2}\vec{B}$$

is the component of $\vec{u}$ perpendicular to $\vec{B}$. So the terms proportional to $\vec{E}$ in the acceleration are

$$\vec{E} - (1 - \cos Bt)\vec{E}_\perp + \frac{\sin Bt}{B}\mathbf{B}\vec{E}$$

$$= \vec{E}_\parallel + \vec{E}_\perp \cos Bt + \frac{\sin Bt}{B}\vec{E}\times\vec{B}$$

Here,

$$\vec{u}_{\parallel} = \vec{u} - \vec{u}_{\perp} = \frac{\vec{u}\cdot\vec{B}}{B^2}\vec{B}$$

is the component of any vector $\vec{u}$ parallel to $\vec{B}$. The terms proportional to $\vec{v}_0$ in the acceleration are

$$\mathbf{B}\,\vec{v}_0 - (1-\cos Bt)\mathbf{B}\,\vec{v}_0 + \frac{\sin Bt}{B}\mathbf{B}^2\,\vec{v}_0$$

$$= \cos Bt\ \vec{v}_0\times\vec{B} - \vec{v}_{0\perp}B\sin Bt \quad .$$

So the final result for the acceleration is

$$\vec{a} = \vec{E}_{\parallel} + \left(\vec{E}_{\perp} + \vec{v}_0\times\vec{B}\right)\cos Bt$$

$$+ \frac{\sin Bt}{B}\ \vec{E}\times\vec{B} - \vec{v}_{0\perp}B\sin Bt$$

Integrating once to find the velocity gives

$$\vec{v} = \vec{v}_0 + \vec{E}_{\parallel}t + \left(\vec{E}_{\perp} + \vec{v}_0\times\vec{B}\right)\frac{\sin Bt}{B}$$

$$+ \frac{1-\cos Bt}{B^2}\ \vec{E}\times\vec{B} - (1-\cos Bt)\vec{v}_{0\perp}$$

or equivalently

$$\vec{v} = \vec{v}_{0\parallel} + \vec{E}_{\parallel}t + \left(\vec{E}_{\perp} + \vec{v}_0\times\vec{B}\right)\frac{\sin Bt}{B}$$

$$+ \frac{1-\cos Bt}{B^2}\ \vec{E}\times\vec{B} + \cos Bt\,\vec{v}_{0\perp} \quad .$$

Integrating once again yields the final result for the path followed by the particle

$$\vec{x} = \vec{x}_0 + \vec{v}_{0\parallel}t + \frac{\sin Bt}{B}\ \vec{v}_{0\perp} + \frac{1}{2}\vec{E}_{\parallel}t^2$$

$$+ \frac{1-\cos Bt}{B^2}\left(\vec{E}_{\perp} + \vec{v}_0\times\vec{B}\right) + \frac{Bt-\sin Bt}{B^3}\ \vec{E}\times\vec{B} \quad .$$

As a check, let's see what happens when B becomes small. Taking the limit yields

$$\vec{x} \rightarrow \vec{x}_0 + \vec{v}_{0\parallel}t + t\vec{v}_{0\perp} + \frac{1}{2}\vec{E}_{\parallel}t^2 + \frac{1}{2}t^2\,\vec{E}_{\perp}$$

$$= \vec{x}_0 + \vec{v}_0 t + \frac{1}{2}\vec{E}t^2 \quad .$$

This is indeed the usual result obtained from solving the equation of motion for constant force.

An interesting and familiar case occurs when the electric field is zero and the initial velocity is perpendicular to the magnetic field. Let's say that

$$\vec{B} = B\hat{z} \quad , \quad \vec{v}_0 = v_0\hat{y} \quad .$$

Then the position is

$$x - x_0 = \frac{1-\cos Bt}{B} v_0$$

$$y - y_0 = \frac{\sin Bt}{B} v_0 \quad .$$

This is the equation of a circle passing through the point (x_0, y_0). To get back to the correct system of units, we have to replace

$$\vec{E} \to \frac{q}{m} \vec{E} \;,\; \vec{B} \to \frac{q}{m} \vec{B} \quad .$$

Restoring these constants, we obtain

$$x - x_0 = \frac{1-\cos(qB/m)t}{qB/m} v_0$$

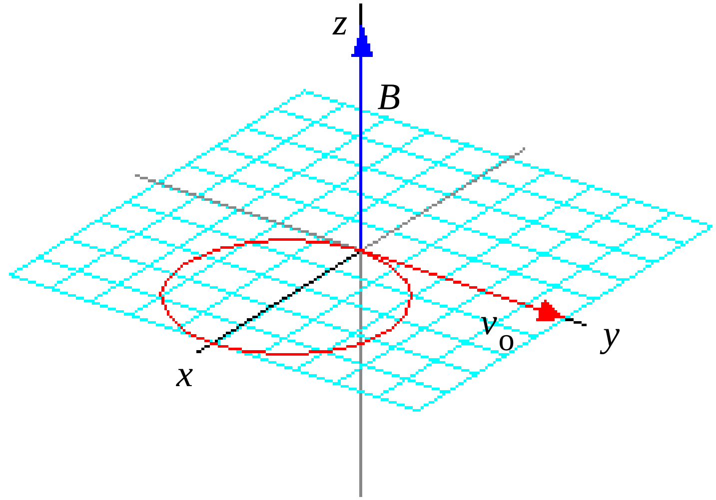
$$y - y_0 = \frac{\sin(qB/m)t}{qB/m} v_0 \quad .$$

We find for the angular frequency and radius of circular motion

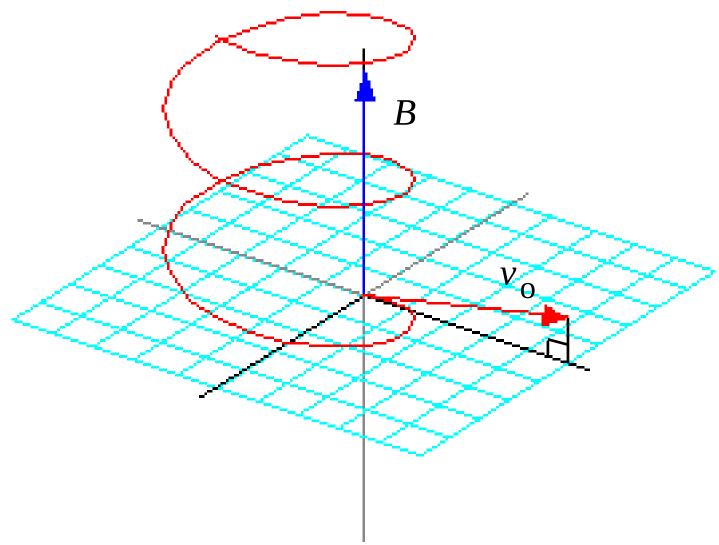
$$\boxed{\omega = \frac{qB}{m} \quad \text{and} \quad R = \frac{v_0}{\omega}} \quad .$$

The former is called the *cyclotron frequency*, which we obtained by more elementary methods elsewhere. The latter is a geometric property of uniform circular motion.

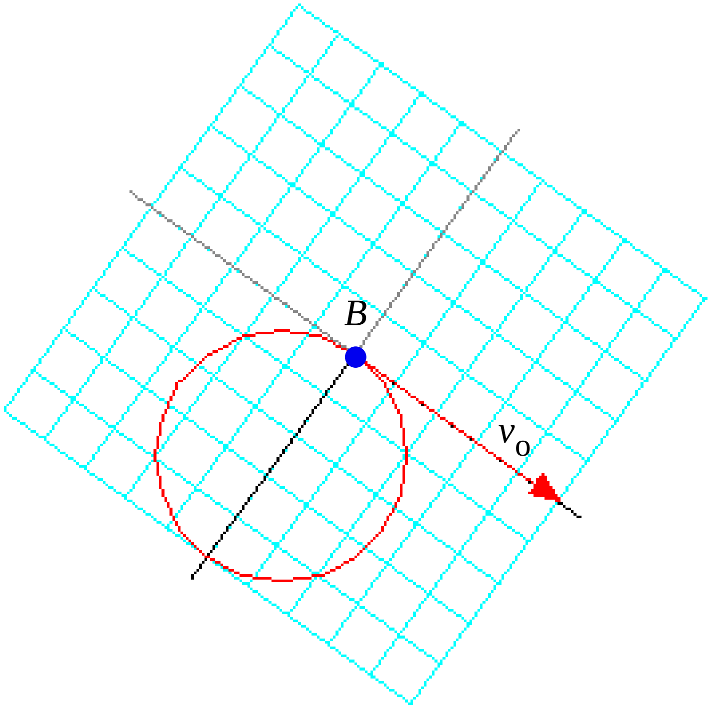
Here is a plot of this path:



If, in addition, the initial velocity has a component parallel to the magnetic field, the path is a spiral, or *helix*.



The projection of the helix onto a plane perpendicular to the magnetic field is a circular path of the type just described. Here is a top view:



Another interesting case occurs when the initial velocity is zero and the electric field is perpendicular to the magnetic field. Let's say that

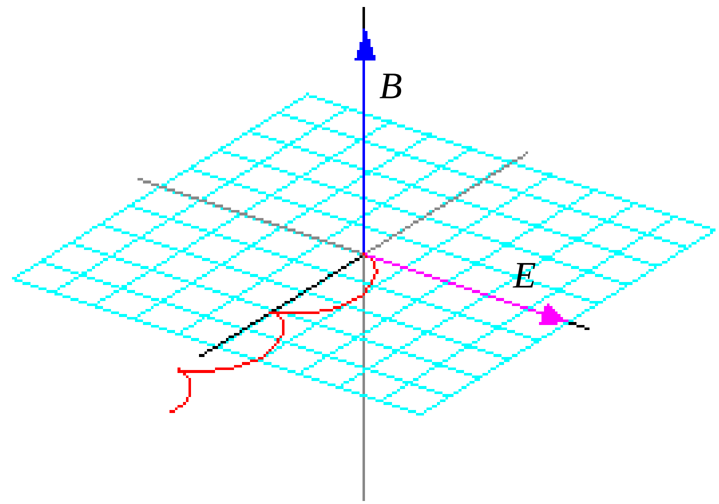
$$\vec{B} = B\hat{z} \quad , \quad \vec{E} = E\hat{y} \quad .$$

Then the position is

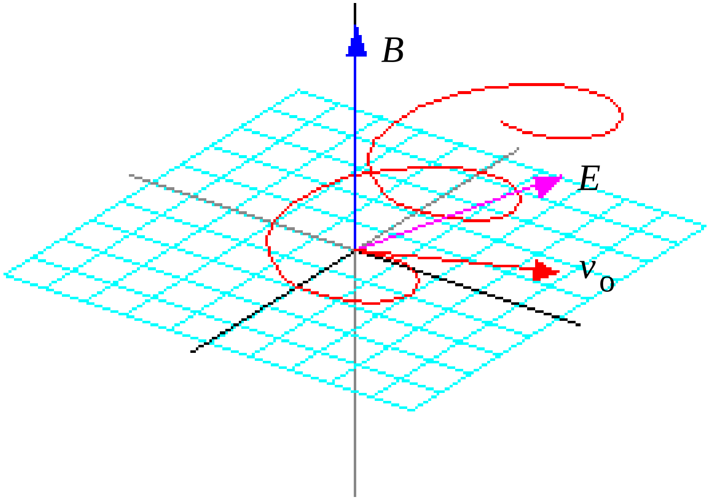
$$x - x_0 = \frac{Bt - \sin Bt}{B^2} E$$

$$y - y_0 = \frac{1 - \cos Bt}{B^2} E \quad .$$

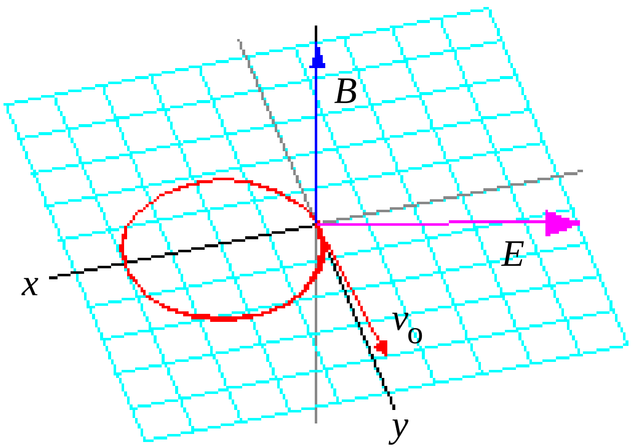
This is the equation of a *cycloid* oriented along the *x* axis. The average velocity in the *x* direction is *E/B*. Here is a plot:



The general motion can be quite complicated. Here is a case of spiral motion in the presence of both magnetic and electric fields. In this case, the projection of the path along its axis is elliptical. (Exercise: calculate the major and minor axes of the ellipse in terms of the fields and the velocity.)



Here is a view along the path's axis:



There is a Java applet which allows you to set the fields and the velocity, and then animates the particle's path. The view angle can be varied. The above figures were made using this applet.