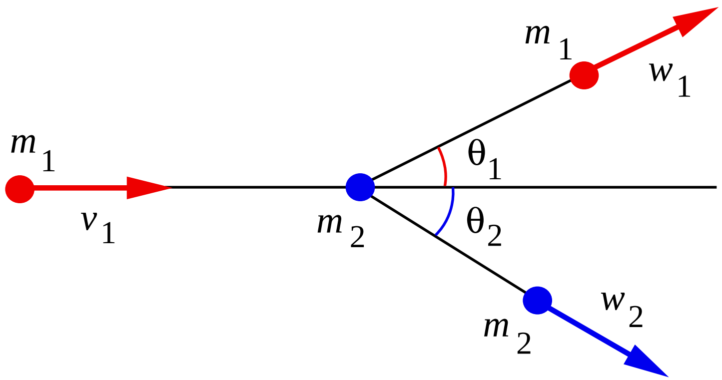
Elastic Collisions

Definition:

• two point masses on which no external forces act collide without breaking up, sticking together, or losing any energy.



Prerequisites:

• collisions in one dimension

• conservation of momentum and energy

Why study it?

• illustrates conservation laws

• illustrates center of mass frame

• occurs frequently in everyday applications

Summary:

The scattering angles and the target's final speed are given by

$$\boxed{\cos\theta_1 = \frac{(1+\alpha)\beta^2 + 1 - \alpha}{2\beta}\,, \qquad \frac{w_2}{v_1} = \sqrt{\frac{1-\beta^2}{\alpha}} \quad \text{and} \quad \cos\theta_2 = \frac{1+\alpha}{2}\,\frac{w_2}{v_1}}$$

where

$$\alpha = \frac{m_2}{m_1} \quad \text{and} \quad \beta = \frac{w_1}{v_1}\,.$$

Go to derivation.

Go to Java™ applet

Momentum and conservation laws

The concept of momentum in three dimensions is not materially different from that in one dimension. The *momentum* vector of a body is its mass times its velocity vector:

$$\boxed{\vec{p} = m\vec{v}} \quad .$$

Suppose we have a *many-body system*. In such systems, the concept of momentum becomes particularly useful. Let's see why.

In general, two kinds of forces can act on the bodies in the system. There are internal forces *between* the bodies, and external forces acting on *individual* bodies.

In what follows, we will make a specific assumption about the nature of the forces acting *between* bodies. We will assume that they satisfy

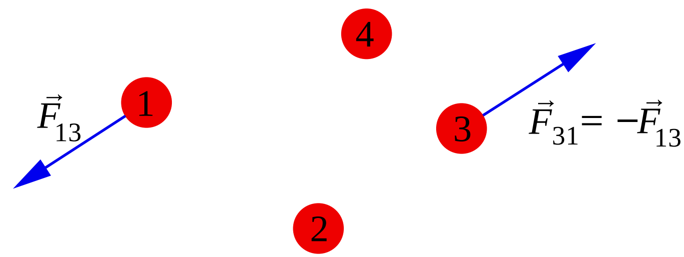
Newton's "third law":

The force of body *j* on body *i* is equal in magnitude and opposite in direction to the force of body *i* on body *j*.

(Here, *i* and *j* stand for numbers which label the bodies.) This means that if the subsystem containing only those two bodies were placed in isolation (no external forces acting), then no net force would act on it.

Although this may seem obvious, *it does not hold for all types of force*. Newton's third law is not a law at all; it is just a description of a "nice" kind of force.

We'll symbolize the force on body *i* due to body *j* by $\vec{F}_{ij}$. Then the third law looks like this in pictures:



and like this in symbols:

$$\vec{F}_{ij} = -\vec{F}_{ji} \quad .$$

We'll assume that a body exerts no force on itself, so $\vec{F}_{ii} = 0$. (This is contained in the above equation.)

Let's also let $\vec{F}_i$ be the external force acting on body *i*. We make no assumptions about its nature. Then Newton's second law applied to body *i* reads

$$m_i \vec{a}_i = \vec{F}_i + \sum_j \vec{F}_{ij} \ .$$

The right-hand side is the sum of the external force and all the internal forces acting on body *i*.

Now let's consider the *total* momentum of the system; it is the sum of the individual momenta:

$$\vec{p}_{tot} = \sum_i m_i \vec{v}_i \ .$$

How does the total momentum change in time?

$$\frac{d}{dt} \sum_i m_i \vec{v}_i = \sum_i m_i \vec{a}_i = \sum_i \vec{F}_i + \sum_{i,j} \vec{F}_{ij} \ .$$

The second term on the right-hand side is zero because the forces, taken in pairs, cancel out by Newton's "third law". Hence, we find

$$\boxed{\frac{d\vec{p}_{tot}}{dt} = \vec{F}_{ext}} \ ,$$

where the sum of the external forces acting on the system is

$$\vec{F}_{ext} = \sum_i \vec{F}_i \ .$$

The internal forces have no effect on the rate of change of the total momentum. Now let's suppose that the total force acting on the system is zero. Then we have a *conservation law*:

$$\frac{d\vec{p}_{tot}}{dt} = 0 \ .$$

This is not as trivial as it might seem, because it is true *even when the internal forces are nonzero*. The only thing that can change the total momentum is a nonzero total *external* force on the system.

The center of mass frame

Suppose you have an object made up of several smaller bodies connected together somehow. You toss this composite body into the air, possibly giving it some spin and giving the consituents some nontrivial internal motions. The resulting motion can be rather complicated. As is often done in physics, we would like to find some aspect of the net motion which is simple.

You have probably already guessed what this aspect is, especially after having studied the center of mass in one dimension. The whole thing carries over very easily into three dimensions, the only change being that many quantities become vectors.

We return to our equation for the rate of change of the total momentum of the system. Written in terms of the position vectors, it reads

$$\frac{d^2}{dt^2}\sum_{\rm i} m_{\rm i}\vec{r}_{\rm i} = \vec{F}_{\rm ext} \quad .$$

We can make this look like Newton's law for a *single* body whose total mass is the sum of all the masses in the system, by multiplying and dividing by the total mass *M*:

$$\boxed{M\frac{d^2\vec{r}_{\rm cm}}{dt^2} = \vec{F}_{\rm ext} \quad \text{where} \quad \vec{r}_{\rm cm} = \frac{1}{M}\sum_{\rm i} m_{\rm i}\vec{r}_{\rm i}} \quad .$$

The quantity $\vec{r}_{\rm cm}$ is called the *center of mass*. It is the average of the positions of all the bodies, weighted by their masses.

The above two equations say that the motion of the center of mass of the system is the same as the motion of a single body

- whose mass is equal to the total mass of the system;
- which is located at the center of mass of the system; and
- which is acted on by a force equal to the sum of the external forces acting on the bodies of the system.

In general, the center of mass moves as time goes on. The velocity of the center of mass is just the time derivative of $\vec{r}_{\rm cm}$, which is

$$\vec{v}_{\rm cm} = \frac{1}{M}\sum_{\rm i} m_{\rm i}\vec{v}_{\rm i}$$

Now, let's do a trick. We will view the whole system from a frame of reference which is moving along with velocity equal to the velocity of the center of mass.

Let's say that a body is moving at 10 m s^{-1} in the original reference frame, and that the center of mass is moving in the same direction at 6 m s^{-1}. Then the velocity of the body when viewed in the center of mass frame is 4 m s^{-1}. To get the velocity in the center of mass frame, you just subtract $\vec{v}_{\rm cm}$.

We will symbolize quantities in the center of mass frame by putting primes on them. Then the equation which connects velocities in the center of mass frame with those in the lab frame is

$$\vec{v}'_{\rm i} = \vec{v}_{\rm i} - \vec{v}_{\rm cm} \; .$$

The momentum of a body in the center of mass frame is just

$$\vec{p}'_{\rm i} = m_{\rm i}\vec{v}'_{\rm i} = m_{\rm i}(\vec{v}_{\rm i} - \vec{v}_{\rm cm}) \quad .$$

Now for the punch line. We can show that *the sum of the momenta in the center of mass frame* *is zero:*

$$\begin{aligned}\sum_i \vec{p}_i^{\,\prime} &= \sum_i m_i(\vec{v}_i - \vec{v}_{cm}) \\ &= \sum_i m_i \vec{v}_i - M\vec{v}_{cm} \\ &= 0 \ .\end{aligned}$$

The last line follows directly from the definition of the velocity of the center of mass.

Energy and the center of mass frame

An important property of the center of mass frame is that

the total kinetic energy of a system is equal to the kinetic energy of the center of mass plus the sum of the kinetic energies of the bodies of the sytem, as calculated in the center of mass frame.

The proof is just a short calculation. Using the velocity transformation law

$$\vec{v}_i = \vec{v}_{cm} + \vec{v}_i^{\,\prime} \ ,$$

and the fact that the sum of the momenta in the center of mass frame is zero:

$$\sum_i m_i \vec{v}_i^{\,\prime} = 0 \ ,$$

we find

$$\begin{aligned}\sum_i \frac{1}{2} m_i \vec{v}_i^{\,2} &= \sum_i \frac{1}{2} m_i \left(\vec{v}_{cm}^{\,2} + 2\vec{v}_{cm} \cdot \vec{v}_i^{\,\prime} + \vec{v}_i^{\,\prime\,2}\right) \\ &= \frac{1}{2}\left(\sum_i m_i\right)\vec{v}_{cm}^{\,2} + \vec{v}_{cm} \cdot \sum_i m_i \vec{v}_i^{\,\prime} + \sum_i \frac{1}{2} m_i \vec{v}_i^{\,\prime\,2} \\ &= \frac{1}{2} M \vec{v}_{cm}^{\,2} + \sum_i \frac{1}{2} m_i \vec{v}_i^{\,\prime\,2} \quad .\end{aligned}$$

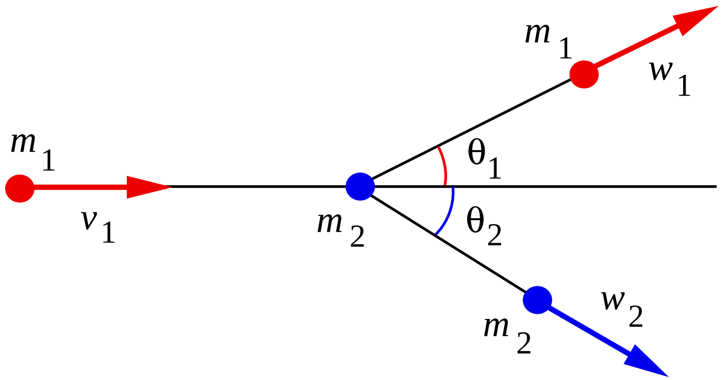
The first term on the last line is the kinetic energy of the center of mass. Another way to write the last equation is

$$T = T_{cm} + T' \ ,$$

where T stands for kinetic energy.

Scattering in two dimensions

Let's apply conservation of momentum and energy to a non-trivial problem: the elastic scattering of one body off another (*elastic* means no energy is lost in the collision).



We will suppose body 2 (the *target*), is initially at rest.

Conservation of energy

$$\frac{1}{2}m_1v_1^2 = \frac{1}{2}m_1w_1^2 + \frac{1}{2}m_2w_2^2 \quad .$$

Conservation of momentum parallel to v_1

$$m_1v_1 = m_1w_1\cos\theta_1 + m_2w_2\cos\theta_2 \quad .$$

Conservation of momentum perpendicular to v_1

$$m_1w_1\sin\theta_1 = m_2w_2\sin\theta_2 \quad .$$

Without loss of generality, we may take w_1 and w_2 to be positive, and the scattering angles to lie between 0 and π.

You may recall our earlier treatment of a collision in one dimension. There, there were two equations in two unknowns, the final speeds. The final speeds were therefore fixed, once the initial speeds were known.

Here, we have *three* equations in *four* unknowns, the two final speeds and the two scattering angles. Therefore, we won't be able to solve for them all. Let's regard w_1 as a variable, and solve for the other three in terms of w_1.

Let's solve for θ_1 (leaving θ_2 and w_2 as exercises). We do this in several steps, which we won't reproduce here.

1) Eliminate θ_2 from the two momentum equations by solving them for $\cos\theta_2$ and $\sin\theta_2$, and then using the identity $\cos^2\theta_2 + \sin^2\theta_2 = 1$.

2) Eliminate w_2 by solving the energy equation for w_2^2 and substituting it into the result of the first step.

3) Solve the resulting equation for $\cos\theta_1$. The result is

$$\boxed{\cos\,\theta_1 = \frac{(1+\alpha)\beta^2 + 1 - \alpha}{2\beta}} \quad ,$$

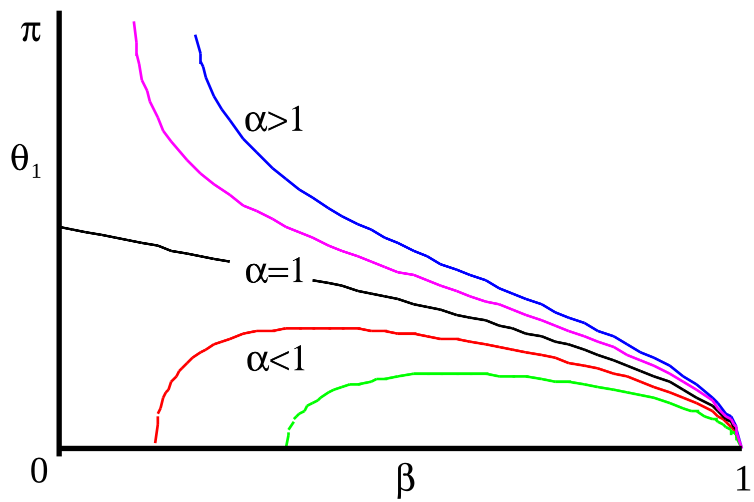
where we have introduced two abbreviations:

$$\alpha = \frac{m_2}{m_1} \quad ,$$

which tells the relative sizes of the two masses, and

$$\beta = \frac{w_1}{v_1} \quad ,$$

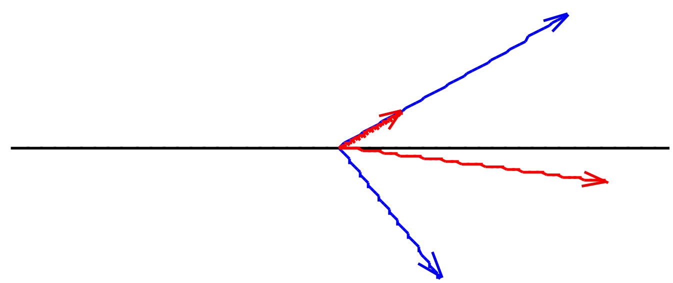
which just gives w_1 in units of v_1. Here is a plot of θ_1 for various values of the mass ratio:



We see several interesting points. First, if $\alpha<1$ (target mass less than projectile mass, an uncommon situation) then there is a maximum of the scattering angle θ_1. It is easy to show that it is given by

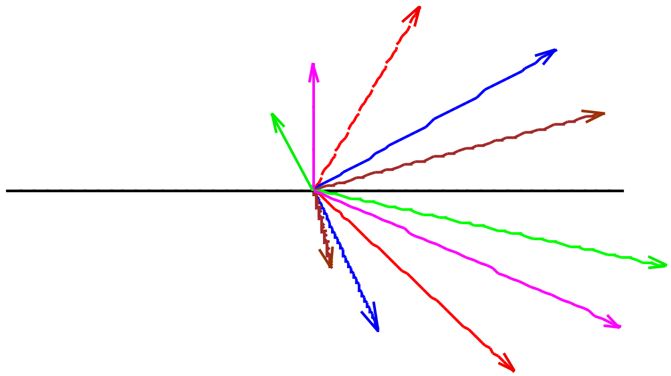
$$\boxed{\sin\,\theta_1^{\mathrm{max}} = \alpha} \quad .$$

For scattering through angles less than this, there are actually *two* values of the final speed for a given value of θ_1. The next figure shows the momenta in a typical case ($\alpha=1/\sqrt{2}$, $\theta_1=\pi/6$). The two red lines correspond to one solution, and the two blue lines correspond to the other.



Here is a movie of the process represented by the two red lines, and another of the one represented by the two blue ones.

On the other hand, if $\alpha>1$ (target more massive than projectile) then scattering through any angle is possible, and there is only one value of the final speed for a given value of θ_1. The following graph shows the momenta in several cases:



Here is a movie of the process represented by the two red lines, and another movie of the process shown in green. Links to movies showing the same processes in the center of mass frame will be provided later.

Another interesting point is that for all values of α, there is a minimum value of the final speed of the projectile. It is given by

$$\boxed{\frac{w_1^{\min}}{v_1} = \frac{|m_1 - m_2|}{m_1 + m_2}} \quad .$$

This minimum occurs at $\theta_1=0$ when $\alpha<1$ and at $\theta_1=\pi$ when $\alpha>1$. It corresponds to a maximum amount of kinetic energy transferred to body 2:

$$\boxed{T_2^{\max} = \frac{4m_1 m_2}{(m_1+m_2)^2}\, T_1} \quad ,$$

where T_1 is the energy of the incoming body 1. In cases where the incoming mass m_1 is much less than the target mass m_2 (as often happens in fixed-targed particle physics experiments), the factor multiplying T_1 on the right-hand side is much less than 1, and not very much of the incident particle's energy can be transferred to the target.

On the other hand, if the two masses are equal then the final velocity of the projectile can be zero. In that case, the factor multiplying T_1 on the right-hand side is equal to 1 and *all* the energy is transferred to the target.

Exercise:

Show that the speed w_2 and angle θ_2 of the target after the collision are given by

$$\boxed{\frac{w_2}{v_1} = \sqrt{\frac{1-\beta^2}{\alpha}} \quad \text{and} \quad \cos\theta_2 = \frac{1+\alpha}{2}\,\frac{w_2}{v_1}} \quad .$$

Analysis in the center of mass frame

In our section on the center of mass frame, we show that this frame is special because the sum of the momenta in this frame is zero. This allows us to simplify the analysis considerably. In the following, we'll denote momenta in the center of mass frame by primes.

There are only two momenta before the collision, so they are equal and opposite in the center of mass frame:

$$\vec{p}_2{}' = -\vec{p}_1{}' \ .$$

There are only two momenta after the collision (we'll call them $\vec{q}_1$ and $\vec{q}_2$), which are also equal and opposite to one another in the center of mass frame:

$$\vec{q}_2{}' = -\vec{q}_1{}' \ .$$

We now apply conservation of energy in the center of mass frame:

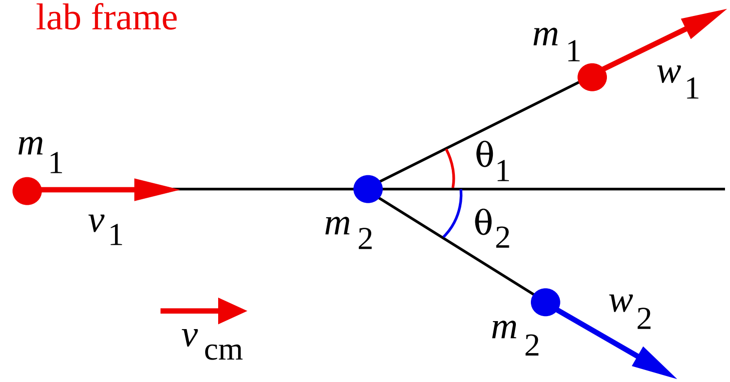
$$\frac{\vec{p}_1{}'^2}{2m_1}+\frac{\vec{p}_2{}'^2}{2m_2}=\frac{\vec{q}_1{}'^2}{2m_1}+\frac{\vec{q}_2{}'^2}{2m_2} \ .$$

We thus find that the magnitudes of the momenta in the center of mass frame do not change during the collision:

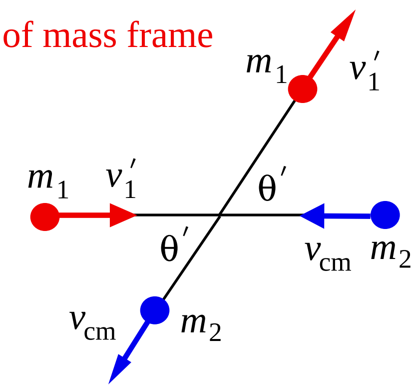
$$\left|\vec{q}_1{}'\right| = \left|\vec{p}_1{}'\right| \qquad \text{and} \qquad \left|\vec{q}_2{}'\right| = \left|\vec{p}_2{}'\right| \ .$$

Since the masses of the bodies don't change during the collision, we therefore conclude that the *speeds don't change*, when viewed in the center of mass frame. Only the direction of motion changes. Here are the diagrams:

lab frame



center of mass frame



In labeling the second diagram, we have already used the facts that the speed of the target is v_{cm}, that the final speeds are the same as the initial speeds, and that the final velocities are back-to-back in the center of mass frame. Much of our work is already done.

The final speeds do not depend on the scattering angle θ' in the center of mass frame. The angular dependence of the final speeds in the lab frame comes in when we transform between the two frames. We do this using the velocity transformation equation

$$\vec{w}_1 = \vec{w}_1' + \vec{v}_{cm} .$$

Writing this out in components and using the fact that $w_1' = v_1'$, we get

x-component of final velocity of projectile:

$$w_1 \cos\theta_1 = v_1' \cos\theta' + v_{cm}$$

y-component of final velocity of projectile:

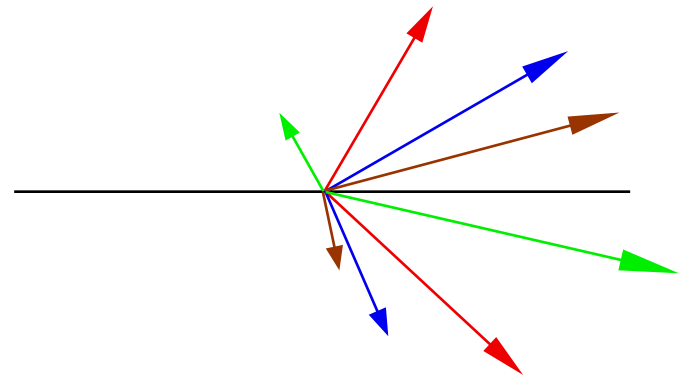
$$w_1 \sin\theta_1 = v_1' \sin\theta'.$$

We easily calculate from their definitions

$$v_{cm} = \frac{m_1 v_1}{m_1 + m_2} \quad \text{and} \quad v_1' = \frac{m_2 v_1}{m_1 + m_2} .$$

We solve the two velocity transformation equations for $\cos\theta'$ and $\sin\theta'$, and then eliminate θ' using the identity $\cos^2\theta' + \sin^2\theta' = 1$. Solving the resulting equation for $\cos\theta_1$ yields the same result as before.

Here are the momenta of some typical scatterings as seen in the lab frame:



In order to compare these processes in the two frames, here is a movie of the process represented by the two red lines as seen in the lab frame, and another of the same process as seen in the center of mass frame. And here is a movie of the process shown in green in the lab frame, and another in the center of mass frame.