Moments of Inertia

Suppose a body is moving on a circular path with constant speed. Let’s consider two quantities: the body’s angular momentum $L$ about the center of the circle, and its kinetic energy $T$. How are these quantities related to its angular velocity $\omega$?

The angular momentum about the center of the circle has magnitude

$$L = mvr,$$

and the velocity has magnitude

$$v = r\omega.$$

The first relation we seek is therefore

$$L = mr^2\omega.$$

This is of the form

\[\text{angular momentum} = \text{constant} \times \text{angular velocity},\]

and reminds us of the analogous equation for linear momentum

$$p = mv,$$

which is of the form

\[\text{linear momentum} = \text{mass} \times \text{linear velocity}.\]

The kinetic energy of the body is

$$T = \frac{1}{2}mv^2,$$

and is of the form

\[\text{kinetic energy} = (1/2) \text{mass} \times \text{linear velocity}^2.\]

Substituting in for the velocity, we find

$$T = \frac{1}{2}mr^2\omega^2,$$

which is of the form

\[\text{kinetic energy} = (1/2) \text{constant} \times \text{angular velocity}^2.\]

The “constant” in these equations is the rotational analog of mass. It is called the moment of inertia of the body about the point at the center of the circle, and is symbolized by $I$:
\[ L = I \omega \quad \text{and} \quad T = \frac{1}{2} I \omega^2. \]

In this particular case, the value of the moment of inertia is

\[ I = mr^2. \]

Now let’s suppose that we have a system consisting of several bodies held rigidly together in a single plane, and that this composite body is rotating about a point in the plane of the body:

Then each piece of the whole thing goes around in a circle, and they all have the same angular velocity. The angular momentum of the system is just the sum of the angular momenta of the parts:

\[ L = \sum_i m_i \mathbf{r}_i^2 \mathbf{\omega}, \]

so the moment of inertia of the system is the sum of the individual moments of inertia:

\[ I = \sum_i m_i r_i^2. \]

In the case of a continuous body, the sum becomes an integral. We will see some cases of this below.

Example: moment of inertia of a ring about its center

Suppose that instead of a single body moving in a circular path, we have a thin ring spinning around on its axis like this:

The mass of the ring is \( m \) and its radius is \( R \). What is the moment of inertia of the ring about its center?

We imagine the ring split up into tiny pieces. All are at the same distance \( R \) from the center of the circle. The expression for the moment of inertia simplifies, becoming

\[ I = \sum_i m_i R^2. \]
The sum of the masses of the pieces is just $m$, so we get the result

$$I = mR^2.$$  

This is the same as the moment of inertia of a single body at distance $R$ from the center.

**Example: moment of inertia of a disc about its center**

Now let’s soup up our example even more. Suppose we have a circular disc spinning around on its axis:

![Diagram of a disc](image)

Let’s let the mass of the disc be $m$, and its radius be $R$. We’ll also suppose that the mass is uniformly distributed over the disc. What is the moment of inertia of the disc about its center?

Well, we can think of the disc as being made up of a bunch of thin rings. We can “add up” the moments of inertia of all the rings using calculus, and the result will be the moment of inertia of the disc.

Let’s see how this works. Consider a typical ring, of radius $r$ and (infinitesimal) thickness $dr$:

![Diagram of a ring](image)

(The thickness has been exaggerated in the figure.) What’s the mass of this ring? Well, the mass will be the mass of the whole disc, times the ratio of the area of the ring to the area of the disc. The area of the ring is just its length times its thickness, or $2\pi r dr$. So the mass is

$$\frac{2\pi r dr}{\pi R^2} m = \frac{2m}{R^2} r dr.$$

The moment of inertia of the ring is its mass times the square of its radius, and we want to add these up for all radii from 0 to $R$. This means we have to do an integral

$$I = \int_0^R r^2 \left(\frac{2m}{R^2} r dr\right) = \frac{2m}{R^2} \int_0^R r^3 dr = \frac{2m}{R^2} \frac{1}{4} R^4.$$

The result is
\[ I = \frac{1}{2} m R^2 \].

This is an interesting result. If all the mass were concentrated at the edge (that is, if the plate were actually a ring), then the moment of inertia would be \( mR^2 \). The moment of inertia of the disc is actually only half as big as this, because the rings nearer to the center contribute less than they would if they were right at the edge.

**Example illustrating dependence of moment of inertia on the point of rotation**

Suppose we return to our thin ring of mass \( m \) and radius \( R \), and we constrain it to rotate about a point on the ring. Here is a top view:

What’s the moment of inertia now?

Well, the principle is no different from before. Only the details of its application change. We just take all the little pieces of the ring and multiply their mass times the square of their distance from the point of rotation. We then add these up over the whole ring.

Warning! This is the kind of calculation that looks very easy when done by someone else, but can appear nearly impossible when you try it yourself! Don’t be fooled! Practice on bodies of other shapes yourself.

One of the biggest challenges when doing a calculation of this kind is the very first step: the choice of suitable coordinates. Here is a good choice for the present case:

The mass of the arc subtended by the angle \( d\theta \), shown in blue, is the mass of the ring times the ratio of the length of the arc to the circumference of the ring, or

\[
\frac{R d\theta}{2\pi R} m = \frac{m}{2\pi} d\theta .
\]

rotate about this point

The mass of the arc subtended by the angle \( d\theta \), shown in blue, is the mass of the ring times the ratio of the length of the arc to the circumference of the ring, or

\[
\frac{R d\theta}{2\pi R} m = \frac{m}{2\pi} d\theta .
\]
The distance of the little arc to the point about which we are calculating the moment of inertia is 
\[ r = 2R \cos \frac{\theta}{2} \].

Multiplying the mass times the square of the distance, and integrating over the whole ring, gives for the moment of inertia
\[ I = \int_0^{2\pi} \left( 4R^2 \cos^2 \frac{\theta}{2} \right) \left( \frac{m}{2\pi} \right) d\theta = \frac{2mR^2}{\pi} \int_0^{2\pi} \frac{1}{2} (1 + \cos \theta) d\theta \]

The value of the integral is \( \pi \), so the result is
\[ I = 2mR^2 \].

Notice that this is the same body for which we earlier calculated the moment of inertia to be half as large! That’s because the two moments of inertia are taken about different points. The moment of inertia is not an intrinsic property of the body, but rather depends on the choice of the point around which the body rotates.

Exercise: moment of inertia of a wagon wheel about its center

Consider a wagon wheel made up of a thick rim and four thin spokes. The rim’s outer radius is \( a \) and inner radius is \( b \), and its mass \( M \) is uniformly distributed. The spokes have length \( b \) and each has mass \( m \), again uniformly distributed.

Show that the moment of inertia of the wheel about its axle is
\[ I = \frac{1}{2} M(a^2 + b^2) + \frac{4}{3} mb^2 \].

Hint: this exercise is designed to be rather complicated. You have to break up the wheel into separate parts, calculate their moments of inertia individually, and add them up in the end.

Example of use of energy of rotation: body rolling down slope

Here is a famous example. Consider a circular body which rolls without slipping down an inclined plane. Let’s not make any assumptions about the nature of the body, except that its mass is \( m \) and its moment of inertia about its central axis is \( I \). What is the acceleration along the slope?
You could tackle this by trying to figure out all the forces involved, but by far the easiest way is to use the method of energy.

Let the distance travelled along the slope be $x$. Then the vertical distance travelled in the downward direction is $x \sin \theta$.

The total energy is

$$E = \frac{1}{2} mv^2 + \frac{1}{2} I \omega^2 - mgx \sin \theta.$$  

The first term is the kinetic energy due to the motion of the center of mass and the second is the kinetic energy due to rotation about the center of mass. (We used a result of an earlier section to split up the kinetic energy in this way.) The third term is the gravitational potential energy.

A little reflection will convince you that if no slipping occurs, then the angular frequency of rotation is related to the speed along the plane by

$$\omega = \frac{v}{R}.$$  

Substituting, we find

$$E = \frac{1}{2} m \left( 1 + \frac{I}{mr^2} \right) v^2 - mgx \sin \theta.$$  

We know that this is constant in time, so we may set its time derivative to zero:

$$0 = mv \left( 1 + \frac{I}{mr^2} \right) a - mgv \sin \theta.$$  

Solving for the acceleration, we find

$$a = \frac{g \sin \theta}{1 + \frac{I}{mr^2}}.$$

Notice that this is less than the acceleration of a body sliding down a frictionless plane at the same angle. Why is this? Well, in the present case the force of friction pushes uphill along the slope, causing the body to rotate as it moves. This extra frictional force slows down the body, compared with a body sliding down a frictionless plane.

**Exercise - practice with forces**

Derive the above result by considering the forces involved.
Exercise - how to measure the moment of inertia

Think of a way in which you could use the above result to measure the moment of inertia of a body with circular cross-section.

The inertia tensor

Up until now, we have been considering only the rotation of two-dimensional planar bodies about axes perpendicular to the body. But you can easily see that this is only a very small subset of all possible rotations of a rigid body. So now let’s move on to the more general case of the rotation of a three-dimensional rigid body about an arbitrary axis. We will assume only that the direction of the axis of rotation does not change.

Let’s let the angular velocity vector of the body be \( \vec{\omega} \). This is the same for every piece of the body. As we learned in our section on rotational motion, the velocity of any particular piece is

\[ \vec{v}_i = \vec{\omega} \times \vec{r}_i. \]

Let’s ask the question: what is the relation between the total angular momentum of the body and the angular velocity vector? We begin by writing down the expression for the angular momentum and substituting for the velocity:

\[ \vec{L} = \sum_{i} \vec{r}_i \times m_i \vec{v}_i = \sum_{i} m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i). \]

We then make use of the standard relation for the triple cross product

\[ \vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}. \]

We find the result

\[ \vec{L} = \sum_{i} m_i \left( r_i^2 \vec{\omega} - (\vec{r}_i \cdot \vec{\omega}) \vec{r}_i \right). \]

This is a nice compact form of the result, but there’s another form that gives more physical insight into the relation between the angular momentum and the angular velocity. This other form is what we get when we re-write the above equation in matrix notation.

Let’s begin by looking at the components of the last equation. We write the position vector as

\[ \vec{r}_i = \hat{x}_i x_i + \hat{y}_i y_i + \hat{z}_i z_i, \]

and obtain for the \( x \)-component of the angular momentum

\[ L_x = \sum_{i} m_i \left( r_i^2 \omega_x - (x_i \omega_x + y_i \omega_y + z_i \omega_z) x_i \right), \]

with similar equations for the other components.

We can re-write this as a linear combination of the three components of the angular velocity:
We recognize the right-hand side as the top component of the product of a matrix and the angular velocity vector:

\[
L_x = \left( \sum_i m_i (r_i^2 - x_i^2) \right) \omega_x - \left( \sum_i m_i x_i y_i \right) \omega_y - \left( \sum_i m_i x_i z_i \right) \omega_z
\]

(For clarity, the other components have been temporarily suppressed.) This matrix is called the \textit{inertia tensor}. We will symbolize it by \( \mathbb{I} \).

Written out in full, it is

\[
\mathbb{I} = \begin{bmatrix}
\left( \sum_i m_i (r_i^2 - x_i^2) \right) & \left( - \sum_i m_i x_i y_i \right) & \left( - \sum_i m_i x_i z_i \right) \\
\left( - \sum_i m_i y_i x_i \right) & \left( \sum_i m_i (r_i^2 - y_i^2) \right) & \left( - \sum_i m_i y_i z_i \right) \\
\left( - \sum_i m_i z_i x_i \right) & \left( - \sum_i m_i z_i y_i \right) & \left( \sum_i m_i (r_i^2 - z_i^2) \right)
\end{bmatrix}
\]

and the relation between the angular momentum and the angular velocity is written compactly as

\[
\vec{L} = \mathbb{I} \vec{\omega}
\]