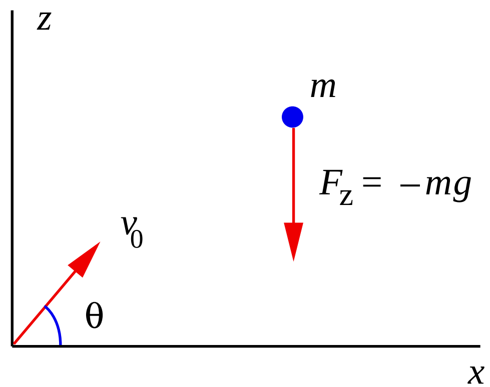
Projectile Motion

Definition:

- body of mass m launched with speed v_0 at angle θ from the horizontal;
- air resistance $\vec{F}_{\text{res}} = -b\vec{v}$, b = nonnegative constant (possibly zero)



Prerequisites:

- fundamentals of Newtonian mechanics
- motion in one dimension, with and without resistance

Why study it?

- a very simple dynamical system with an exact solution in closed form;
- occurs frequently in everyday applications

Summary:

The body's position as a function of time is

$$\boxed{\begin{aligned} x(t) &= \frac{m}{b} v_0 \cos\theta \left(1 - \exp\left(-\frac{b}{m}t\right)\right) \\ z(t) &= -\frac{mg}{b}t + \frac{m}{b}\left(v_0 \sin\theta + \frac{mg}{b}\right)\left(1 - \exp\left(-\frac{b}{m}t\right)\right) \end{aligned}}$$

Go to derivation.

Go to MAPLE code

Go to Java™ applet

We are now going to analyze a standard problem in elementary physics: projectile motion. Suppose we launch a projectile with a fixed speed v_0 at some angle θ from the earth's surface. What path does it follow, and at what angle must we launch it, in order for it to travel the maximum distance along the earth's surface?

For simplicity, we'll first solve the problem in the artificial case in which there is no air resistance, and then we'll go on to the more general case.

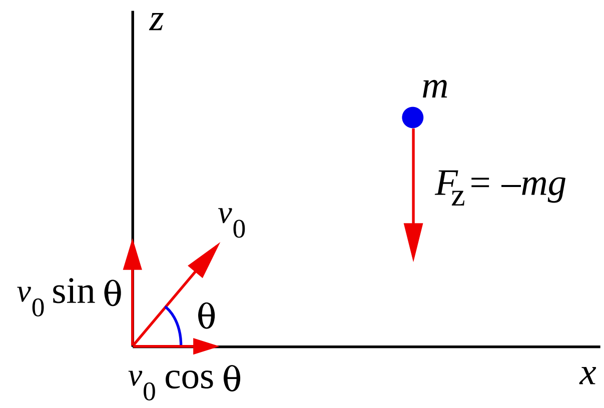
Although we have not yet studied the gravitational interaction, you may be familiar with the relevant fact: near earth's surface, *all* bodies (whatever their inertial mass) have an acceleration of approximately 9.81 m s^{-2} in the downward direction. This is true as long as the effects of air resistance are negligible. This value is given a special symbol, g, and is called the *gravitational acceleration at earth's surface***:**

$$g \approx 9.81 \text{ m s}^{-2} .$$

Let's orient our coordinates so that the z-axis points upwards. Then the force due to gravity is

$$\vec{F} = -mg\hat{z} \ ,$$

where $\hat{z}$ is the unit vector in the z direction. The minus sign indicates that the force is directed downwards. This force is constant - does not depend on time, position, or velocity.



The force of gravity is always in the negative z-direction - it has no component in the x-direction. Accordingly, the problem splits into two separate ones, both of which we have already solved when we considered one-dimensional motion.

z-direction: motion with constant acceleration

$$m\ddot{z} = -mg \ .$$

x-direction: motion with constant velocity

$$m\ddot{x} = 0 \ .$$

(Recall that the double dot notation stands for second time derivative).

We choose coordinates so that the initial values of x and z are both zero. The initial velocities are the components of the velocity vector shown in the last figure:

$$\dot{z}(0) = v_0 \sin\theta \quad \text{and} \quad \dot{x}(0) = v_0 \cos\theta \ .$$

The solutions to the equations of motion with these initial conditions are

$$x(t) = \left(v_0 \cos\theta\right)t \qquad \text{and}$$

$$z(t) = \left(v_0 \sin\theta\right)t - \frac{1}{2} g\, t^2 \ .$$

If we are interested only in the *shape* of the trajectory, we eliminate t from these equations:

$$z = x \tan\theta - \frac{g}{2v_0^2 \cos^2\theta} x^2 \ .$$

This is a parabola opening downwards: a “parabolic arch”. (This expression for z is valid as long as $\theta \neq \pi/2$ - when the projectile is fired vertically. In that case, x remains zero.)

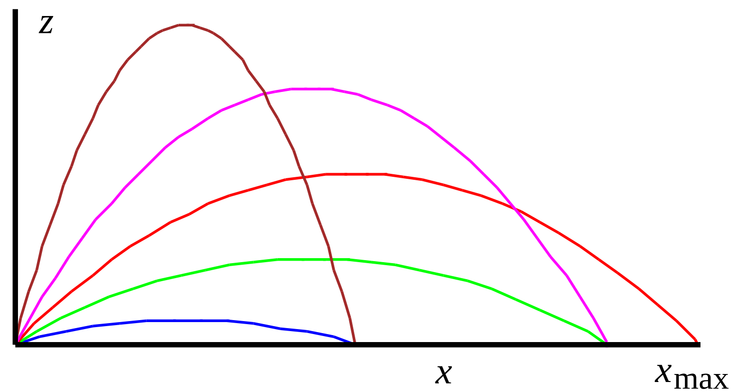
How far does the projectile go, before hitting the ground? To answer this, we must find the value(s) of x for which z=0. Of course, one solution is x=0 - the point of launch, but we are not interested in that one. The other one is easily found to be

$$x(\theta) = \frac{v_0^2 \sin 2\theta}{\mathrm{g}} \ .$$

From this, we find that the maximum distance

$$x_{\mathrm{max}} = \frac{v_0^2}{g} \quad \text{is reached when} \quad \theta = \frac{\pi}{4} \ .$$

Let’s plot the paths for several different values of θ (all with the same initial speed):

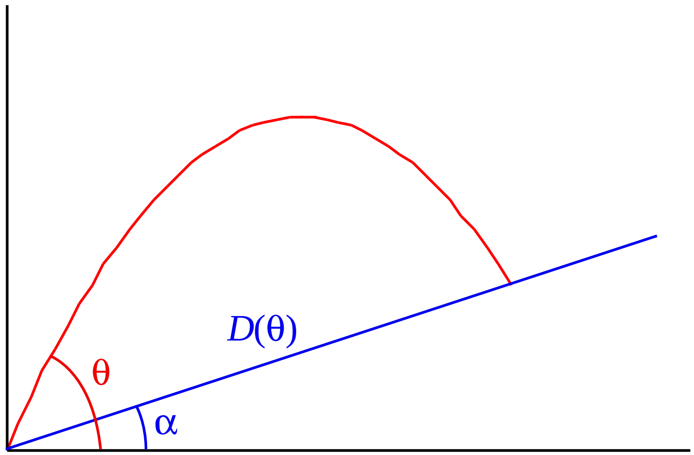


We see that the trajectory that starts out at θ=π/4 - the red one - goes the farthest. Trajectories that start out at smaller θ don’t go as high or as far, and trajectories that start at larger θ go higher, but still not as far.

The resources for this section contain a movie comparing various trajectories. There is also a movie illustrating the fact that the velocity in the horizontal direction is constant, while that in the vertical direction is not.

Exercise: firing projectile up a hill

Suppose you are firing a projectile up a hill. At what angle θ to the horizontal should you fire it, so that it goes the maximum distance along the surface of the hill?



The answer is

$$\theta = \frac{\pi}{4} + \frac{\alpha}{2} \ .$$

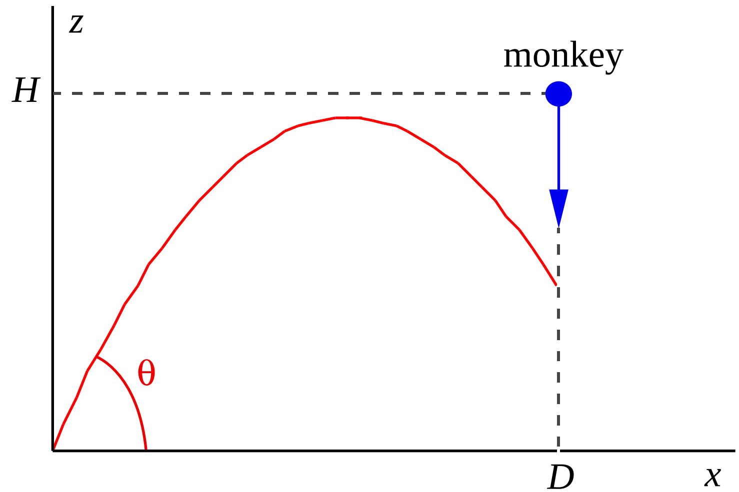
(Hint: show that

$$D(\theta) = \frac{2v_0^2 \cos\theta \sin(\theta - \alpha)}{g \cos^2\alpha} \ .$$

Note that this expression goes over into our previous expression for $x(\theta)$ when $\alpha=0$.)

Example: shooting the monkey

Here's a classic example that every beginning physics student has traditionally learned. A hunter sees a monkey in a tree, and decides to shoot it. He knows that this particular species of monkey always falls from the tree *at the instant the shot is fired*. At what angle must the hunter aim, in order to hit the monkey?



The first step is to solve for the time at which impact occurs. The x-component of the velocity of the projectile is $v_0 \cos\theta$, and the distance in the x-direction is D, so the time of impact is

$$t = \frac{D}{v_0 \cos\theta} \ .$$

At the time of impact, the z-coordinates of the monkey and bullet must be the same:

$$H - \frac{1}{2} g t^2 = (v_0 \sin\theta) t - \frac{1}{2} g t^2 \ .$$

The terms involving the acceleration cancel out (both bodies accelerate at the same rate - Galileo's experiment!), and therefore

$$t = \frac{H}{v_0 \sin \theta} \ .$$

Equating the two expressions for the time of impact yields

$$\tan \theta = \frac{H}{D} \ .$$

This says that *the hunter should aim directly at the monkey*! The downward acceleration of the monkey exactly compensates for the downward acceleration of the bullet, as long as they start falling at the same time.

It may at first seem a bit surprising that this result doesn't depend on the speed of the bullet, but it makes sense when you think about it. The faster the bullet, the less time the bullet has to fall, but the monkey also has less time to fall, so the speed has no effect.

The resources for this section contain a movie illustrating the manner in which the monkey meets his maker. (Disclaimer: no actual monkey died to make this movie.)

Inclusion of air resistance

In many physical cases, there is some resistance to motion. For example, a body could be sliding on a surface with friction present. Or, a body could be moving near the earth, with air resistance.

In many cases, the force resisting the motion is proportional to the *velocity* of the body. Mathematically, this is written as

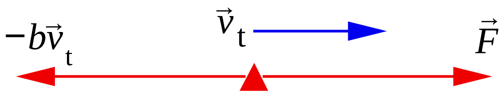
$$\vec{F}_{\text{res}} = -b\vec{v},$$

where b is a positive constant. The minus sign indicates that the force resists the motion, so is directed *opposite* to the velocity.

We would like to illustrate the procedure of solving Newton's law when such a force is involved. Just to make it interesting, we will suppose that the total force on the body is the sum of a constant force $\vec{F}$ and the above resistive force. This will make our analysis immediately applicable to the case of projectile motion in a uniform gravitational field.

It is always a good idea to use physical intuition to get an idea of the nature of the solution, before beginning the mathematics. In the present case, it makes it easy to see one important aspect of the solution without doing a lot of calculations.

As time goes on, the external constant force will just balance the resistive force, giving zero net force. The body will then move with a constant velocity called the *terminal velocity* v_t.



The above figure shows the equal and opposite forces in red. The net force is

$$\vec{F}_{net} = \vec{F} - b\vec{v}_t = 0 \ ,$$

which gives for the terminal velocity

$$\vec{v}_t = \frac{\vec{F}}{b} \ .$$

Let's solve the equation of motion and see how this is reflected in the solution. Newton's law reads

$$m\frac{d\vec{v}(t)}{dt} = \vec{F} - b\vec{v}(t),$$

which we have written entirely in terms of the velocity and its first derivative.

The solution is analogous to the [one-dimensional case](). The result is

$$\vec{v}(t) = \frac{\vec{F}}{b} + \left(\vec{v}_0 - \frac{\vec{F}}{b}\right)\exp\left(-\frac{b}{m}t\right),$$

where $\vec{v}_0$ is the velocity at t=0. As t becomes large, the second term vanishes and the velocity approaches $\vec{F}/b$, as we know it should. Notice that the second term never actually becomes zero at any finite time - it just gets closer and closer.

Integrating once again, we find that the position vector is given by

$$\boxed{\vec{x}(t) = \vec{x}_0 + \frac{\vec{F}}{b}t + \frac{m}{b}\left(\vec{v}_0 - \frac{\vec{F}}{b}\right)\left(1 - \exp\left(-\frac{b}{m}t\right)\right)} \ ,$$

where $\vec{x}_0$ is the position vector at t=0.

Application: projectile motion with air resistance

Let's go back and examine projectile motion, this time including air resistance. With our coordinates oriented in the same way as [before](), the constant force due to gravity is $\vec{F} = -mg\hat{z}$, and we find that the above vector equation gives two separate equations:

$$x(t) = \frac{m}{b}v_0\cos\theta\left(1 - \exp\left(-\frac{b}{m}t\right)\right)$$

$$z(t) = -\frac{mg}{b}t + \frac{m}{b}\left(v_0\sin\theta + \frac{mg}{b}\right)\left(1 - \exp\left(-\frac{b}{m}t\right)\right) \ .$$

Here is some [MAPLE]() code which will generate the above solutions:

```
eq1:=(D@@2)(x)(t)=-(b/m)*D(x)(t):
eq2:=(D@@2)(z)(t)=-g-(b/m)*D(z)(t):
dsolve({eq1,x(0)=0,D(x)(0)=v[0]*cos
(theta)},x(t));
dsolve({eq2,z(0)=0,D(z)(0)=v[0]*sin
(theta)},z(t));
```

If all we are interested in is the *shape* of the trajectory, we should eliminate the time and express z directly in terms of x. Solving for the time from the x-equation, which we can do as long as $\theta \neq \pi/2$, we get

$$t = -\frac{m}{b}\ln\left(1-\frac{bx}{mv_0\cos\theta}\right) .$$

Inserting this into the z-equation, we obtain

$$z = \frac{m^2 g}{b^2}\ln\left(1-\frac{bx}{mv_0\cos\theta}\right) + \left(\sin\theta+\frac{mg}{bv_0}\right)\frac{x}{\cos\theta}.$$

At this point, we begin to realize how tedious it is to keep writing all the constants. It is more efficient to express everything in dimensionless form. To do this, we have to decide on appropriate units.

For our unit of length, it is sensible to take the maximum horizontal distance the projectile can travel with no air resistance; from before, we know this is v_0^2/g. Dividing by this unit, we get dimensionless variables

$$\xi = \frac{g}{v_0^{\,2}}x \quad \text{and} \quad \eta = \frac{g}{v_0^{\,2}}z \ .$$

For a dimensionless quantity that indicates the amount of damping present, it is sensible to take the ratio of the initial speed to the terminal speed. From before, we know that the latter is mg/b, so our dimensionless damping coefficient is

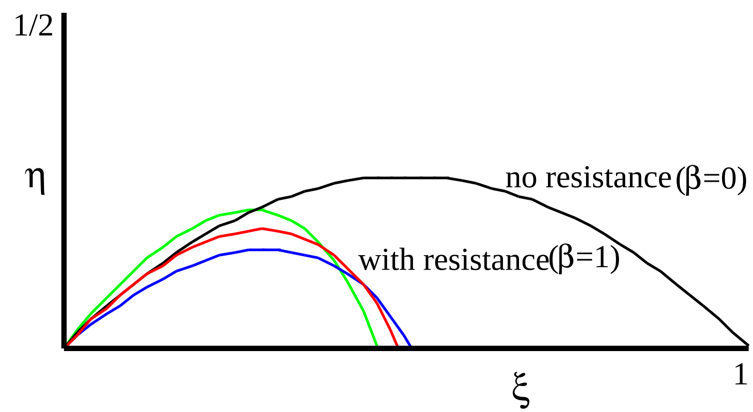
$$\beta \equiv \frac{v_0}{v_t} = \frac{bv_0}{mg} \ .$$

This quantity is zero when no damping is present. In terms of these dimensionless variables, the equation for the shape of the trajectory reads

$$\boxed{\eta = \frac{1}{\beta^2}\ln\left(1-\frac{\beta\xi}{\mathrm{c}}\right) + \left(\mathrm{s}+\frac{1}{\beta}\right)\frac{\xi}{\mathrm{c}}} \ ,$$

where c and s are short for $\cos\theta$ and $\sin\theta$, respectively. This form of the equation is much easier to work with than the original.

It turns out that the maximum horizontal distance is no longer attained when the launch angle is 45 degrees, but rather something *less* than that. In the next plot, the red trajectory has a 45-degree launch angle and the blue one, launched at a smaller angle with the same speed, goes farther:



Also shown for comparison is the 45-degree trajectory with no air resistance. We see how the trajectory with resistance becomes skewed by the decreasing horizontal velocity. The trajectory with no resistance is symmetric (a parabolic arch). We also see that the horizontal distance covered is less when resistance is present. This is what you might expect.

Let's find out *how much* less the distance is. To find how far the projectile goes at a given angle θ, we set $\eta=0$ and solve for ξ. One solution is $\xi=0$, of course, but we are not interested in that one. The other solution is more complicated, and can't be expressed in closed form. In general, it must be obtained numerically.

However, we can obtain an approximate solution in the case where the resistance is small. Then, we can expand in powers of β. (For practice in expansions like this, see the section on techniques for checking answers.) We will keep correction terms linear in β. That means that we must keep terms up to the *cubic* term in the expansion of the logarithm:

$$0 \approx \frac{1}{\beta^2}\left(-\frac{\beta\xi}{c}-\frac{1}{2}\left(\frac{\beta\xi}{c}\right)^2-\frac{1}{3}\left(\frac{\beta\xi}{c}\right)^3\right) + \left(s+\frac{1}{\beta}\right)\frac{\xi}{c} \ .$$

The terms of order $1/\beta$ cancel out, and we may multiply the rest by c/ξ, obtaining

$$0 = s - \frac{\xi}{2c} - \beta\frac{\xi^2}{3c^2} \ .$$

We know from before that the answer when $\beta=0$ is

$$x = \frac{2v_0^2\sin\theta\cos\theta}{g} \ , \quad \text{or} \quad \xi = 2sc \ .$$

This agrees with the equation we just obtained. Let's write the answer when the damping is nonzero as

$$\xi \approx 2sc\,(1+\alpha\beta)$$

and solve for the coefficient α. We find

$$\alpha = -\frac{4}{3}s \ .$$

Therefore, we have the result for the horizontal distance travelled at a given angle θ:

$$\boxed{\xi \approx \sin 2\theta\left(1-\frac{4}{3}\beta\sin\theta\right)}$$

What is the maximum horizontal distance possible, and for what angle is this attained? To find the maximum, we set the derivative to zero:

$$0 \approx 2\cos 2\theta\left(1-\frac{4}{3}\beta\sin\theta\right)-\frac{4}{3}\beta\sin 2\theta\cos\theta .$$

The answer when $\beta=0$ is $\theta=\pi/4$. So we write

$$\theta \approx \frac{\pi}{4} + \varepsilon\beta$$

and solve for the coefficient ε. First, we work out

$$\cos\left(\frac{\pi}{2} + 2\varepsilon\beta\right) = -\sin(2\varepsilon\beta) \approx -2\varepsilon\beta \,.$$

Inserting this and dropping all but linear terms:

$$0 \approx -4\varepsilon\beta - \frac{4}{3}\beta\cos\frac{\pi}{4} \quad .$$

Hence,

$$\varepsilon = -\frac{1}{3\sqrt{2}} \quad .$$

So the maximum distance is attained when the launch angle is

$$\boxed{\theta \approx \frac{\pi}{4} - \frac{\beta}{3\sqrt{2}}} \quad .$$

Note that this is slightly *less* than $\pi/4$, the value when the air resistance is zero. This may or may not be surprising, depending on how good your physical intuition is.

To find out the maximum distance, we insert this back into the expression for ξ. First, we work out

$$\sin\left(\frac{\pi}{2} + 2\varepsilon\beta\right) = \cos(2\varepsilon\beta) \approx 1 \,.$$

Hence, the maximum distance is

$$1 \times \left(1 - \frac{4}{3}\beta\sin\frac{\pi}{4}\right) \ ,$$

and we have a simple result for the maximum horizontal distance attainable:

$$\boxed{\xi_{max} \approx 1 - \frac{4\beta}{3\sqrt{2}}} \quad .$$

Of course, this is less than 1, the value with no resistance.