Vectors in two dimensions

Until now, we have been working in one dimension only. The main reason for this is to become familiar with the main physical ideas like Newton's second law, without the additional complication of vectors. The time has come, however, to take this additional step. Many interesting physical effects are inherently more than one-dimensional, and we would like to learn about them.

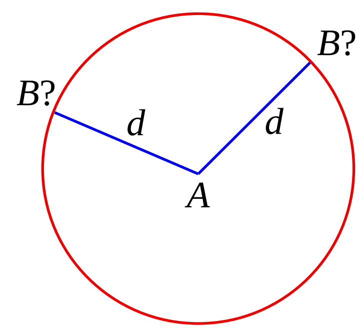
Studies have repeatedly shown that a poor grasp of vectors is one of the major causes of failure in introductory physics courses. The time and effort spent on developing a good understanding of vectors now will be amply rewarded later on.

One of the unfortunate facts about this topic is that students come into first-year physics with widely different levels of background in vectors. Another fact is that university-level professors often don't want to spend much valuable class time reviewing vectors. They prefer to get on to the "real" physics as soon as possible, and if you haven't already magically grasped enough knowledge about vectors, look out!

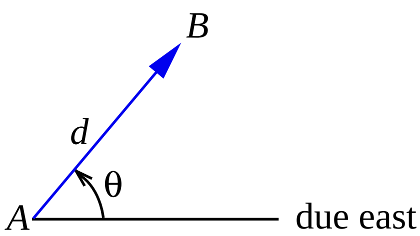
This section, and its companion on vectors in three dimensions, attempts to address these problems. You will find that the emphasis is placed on understanding the concepts involved, while the mathematics has been kept to a minimum. Knowledge of trigonometry is assumed, however. You are really supposed to have learned this in high school. If you are weak in this area, now is the time to review.

What is a vector?

Suppose we are in city A, and someone has told us that a nearby city B is some distance d away. Do we have enough information to find B? Of course not, because B could be at any point on a circle of radius d centered at A. Here is a bird's-eye view:



In addition to the *distance*, we need to know the *direction*, in order to find B. One way of specifying the direction would be to give the angle θ (the Greek letter theta) of B, measured counter-clockwise from due east. Once both the distance and direction of the line from A to B are specified, we can locate B. This is shown in the following figure:



Taken together, the distance and direction of the line from A to B is called the *displacement* from A to B, and is represented by the blue arrow in the above figure. (The arrow-head tells us that we are talking about the displacement from A to B, and not B to A.) The displacement is a classic example of a *vector*.

Definition: a **vector** is a quantity that has both **magnitude** and **direction**.

The magnitude of the displacement vector from A to B is the *distance* from A to B. It is important to remember that a vector is not completely specified by its magnitude or direction alone; *both* are necessary.

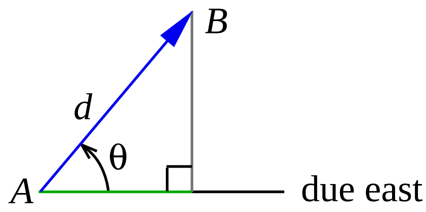
Another familiar example of a vector is the *velocity*. This vector points in the direction of motion, and its magnitude is the *speed*. If only the speed is specified, then the direction is unknown and the velocity is not completely specified.

Baseball pitchers often talk about having "good velocity". Most of the time, what they are really talking about is just the magnitude of the velocity; their word for the ability to determine the direction of a pitch is "control". Often a pitcher will have "good velocity" but "no control". He won't make it in either baseball or physics.

Because of the directional nature of a vector, it looks different when viewed from different directions. In the resources for this lesson you will find a movie illustrating the changing appearance of a vector when viewed from different angles.

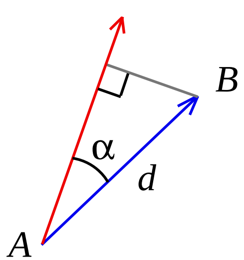
Components of a vector

Returning to the displacement vector from A to B, let's ask: "how much of the displacement is in the easterly direction?" This is the same as asking how far B is from A, when viewed from due south. The answer is the length of the segment shaded in green in the following figure:

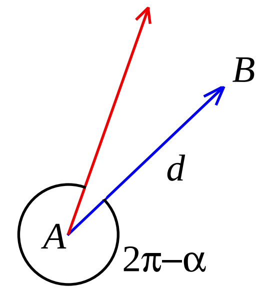


This length is referred to as the *component* of the displacement vector in the easterly direction. It is given in this example by $d \cos \theta$.

There's nothing special about the fact that we found the component of the displacement vector in the easterly direction. We can find its component in any direction we like. We simply specify the direction of interest by drawing a vector in that direction, and drop a line from the tip of the original vector perpendicular to this new vector. For example, the component of the blue vector in the direction of the red vector in the following figure is $d \cos \alpha$:

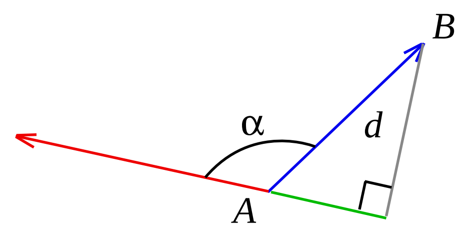


You might think that there is an ambiguity caused by the fact that we could equally well define the angle between the above two vectors in this way:



However, $\cos(2\pi - \alpha)$ is the same as $\cos \alpha$, so there is no ambiguity.

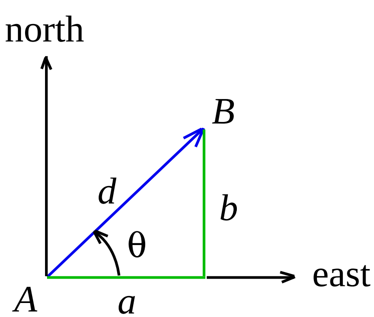
For the value of α shown in the last two figures, $\cos \alpha$ is positive. This is not always the case, however. For example, the red vector could point like this:



The component in the direction of the red vector is still $d \cos \alpha$, but this is now negative. The absolute value of this number is the length of the green segment in the above figure.

Specifying a vector in two dimensions

Although a vector is defined as a quantity with magnitude and direction, it need not be specified directly by the values of these two properties. Returning to our discussion of the displacement vector, we could equally well specify the displacement by saying how far east and how far north B is from A. That is, we could specify the components of the displacement vector in the easterly and northerly directions. These are shown in green in the following figure:



If the vector is specified in this way, it is said to be in *component form*. This is completely equivalent with the *polar form*, in which the vector is specified by its magnitude and its direction.

Of course, you can convert freely back and forth between the two forms. This is a very common manipulation. If you know the polar form (i.e. d and θ are known), then the components a and b are given by

$$a = d\cos\theta \quad \text{and} \quad b = d\sin\theta.$$

Conversely, if a and b are known, then d and θ are given by

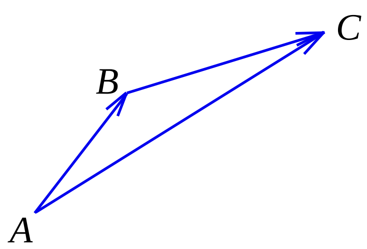
$$d = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \arctan(b/a).$$

You should not memorize these equations. They will occur in many different contexts, with different variables and in different notations. However, you *should* be completely familiar with the *ideas* behind these equations, so that actually performing a conversion between polar and component form presents no problem. The ideas, and not the mathematics, are the primary content of this lesson.

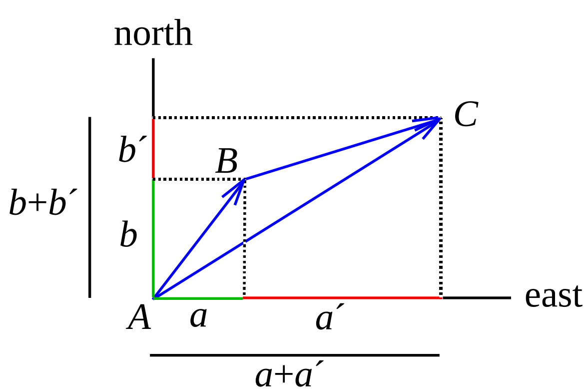
Addition of vectors in two dimensions

Suppose we have a third city, C, and suppose we know the distance and direction from B to C (in addition to our previous knowledge of the displacement vector from A to B). Let's say we want to go directly from A to C. What are the distance and direction?

The first thing to notice is that if the three cities do not lie in a straight line, then the distance from A to C will *not* be equal to the sum of the distances from A to B and from B to C. Also, the direction will be related in a complicated way to the two separate directions and distances:



You can see, however, that the solution is easy if we work with the components of the displacement vectors. Let the components of the vector from B to C in the easterly and northerly directions be be a' and b', respectively. Then it is obvious that the component of the displacement vector from A to C in the easterly direction is $a + a'$, and in the northerly direction is $b + b'$:



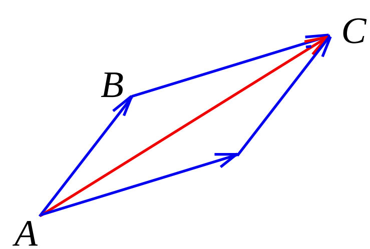
Then the length of the displacement vector from A to C is

$$\sqrt{(a+a')^2 + (b+b')^2}$$

and its angle measured counter-clockwise from due east is

$$\arctan\left(\frac{b+b'}{a+a'}\right).$$

This completely specifies the *sum* of the two separate displacement vectors. To get the sum of two vectors, you place them tip to tail and draw a third vector from the tail to the tip of the whole thing. In the following figure, the sum of the displacement vectors from A to B and from B to C is shown in red.



Note that the *order* in which you do the sum is unimportant. As the above figure shows, adding the vectors in the opposite order requires you to slide them around parallel to themselves. You get what looks like a fictitious path from A to C, going through some ghostly "fourth city". This is all right; what matters is the resulting path directly from *A* to *C*. This is the same, no matter which order the vectors are combined in.

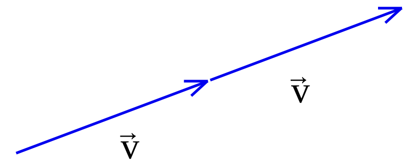
Here is an important point that often causes confusion. A vector is specified completely by its magnitude and direction. The vector is the same, *no matter where it is*, as long as its magnitude and direction are the same. The location of the vector is not part of its definition. You are free to "slide the vectors around" as long as you do not change their magnitude and direction.

Vector notation

Instead of referring to a vector by a name like "the displacement vector from *A* to *B*", it is useful to have a symbol. We denote a vector by an arrow over a letter like this: $\vec{v}$. Different vectors will be distinguished by different letters. The sum of two vectors is written $\vec{u}+\vec{v}$, for example.

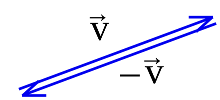
Multiplication of a vector by a real number

Suppose we add $\vec{v}$ to itself. We end up with a vector which is twice as long as the original, pointing in the same direction:



It is natural to write $\vec{v}+\vec{v}=2\vec{v}$, where the right-hand side means a vector in the same direction as $\vec{v}$ but twice as long. Obviously, you can multiply a vector by any positive real number in the same way; for example, $1.5\vec{v}$ is a vector in the direction same direction as $\vec{v}$ but 1.5 times as long.

Suppose we *subtract* $\vec{v}$ from itself. The result is obvious because when you subtract something from itself, you get zero: $\vec{v}-\vec{v}=\vec{0}$. (The right-hand side is the *zero vector*, a vector of length zero whose direction is undefined.) This picture shows the operation of subtracting $\vec{v}$ from itself:



Subtracting $\vec{v}$ is the same as adding a vector the same length as $\vec{v}$ but in the opposite direction. That is,

$$\vec{v}-\vec{v}=\vec{v}+(-\vec{v})$$

Hence, if you multiply a vector by -1, you get a vector the same length as the original but in the opposite direction. Similarly, multiplying a vector by –1.5, say, gives a vector in the opposite direction and 1.5 times as long.

Multiplying a vector by the real number zero obviously gives the zero vector.

It's also useful to have a notation for the length or magnitude of a vector. It is

$$|\vec{v}| \ ,$$

and is a positive number or zero, by definition.

Summary so far

You now know all of the essential information about vectors:

• they have magnitude and direction;

• you can find their component in any direction you choose;

• they can be added together;

• they can be multiplied by a real number;

• there exists a zero vector.

Although we have illustrated the above points using vectors in two dimensions only, everything carries over into three dimensions.

There is a section on vectors in three dimensions in this course material. You don't need to go there now, if all you want to do is understand the concept of vectors, however.

The dot product

The *dot product* (or scalar product, or inner product) of two vectors is defined to be the product of the lengths of the two vectors times the cosine of the angle between the vectors:

$$\vec{a} \cdot \vec{b} = |\vec{a}|\left|\vec{b}\right| \cos\theta \ .$$

The dot product is just a number, in contrast to another kind of product called the *vector product* or *cross product*, to be discussed later.

Note that the length of a vector is just the square root of the dot product of the vector with itself:

$$|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}} \ .$$

Unit vectors

Unit vectors are a handy way to specify directions. Until now, we have specified directions by saying things like “due east” and “due north”. It is often useful to have a shorthand notation for these terms. What we are now going to describe is *just notation* - there is no more content to it than that.

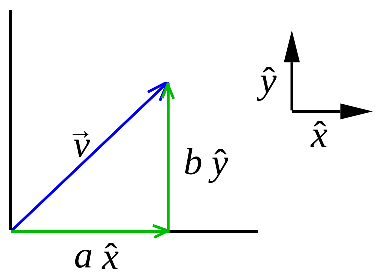
Let's make a vector which has length equal to one unit and points due east. We'll call this the *unit vector in the x-direction* and symbolize it by putting a hat over it:

$$\hat{x} \ .$$

Similarly, let's let the unit vector which points due north be $\hat{y}$.

(Other common notations for these unit vectors are $\hat{i}$ and $\hat{j}$ or $\hat{e}_x$ and $\hat{e}_y$.)

Returning to our diagram which shows the components of a vector in the easterly and northerly directions, we find that the vector can be expressed as the sum of multiples of the unit vectors:



Remember that $a\hat{x}$ is a vector in the $\hat{x}$-direction which has length a. In symbols,

$$\vec{v} = a\hat{x} + b\hat{y} \ ,$$

where a and b are related to the magnitude and direction of $\vec{v}$ as before.

Vectors are particularly easy to manipulate when written like this. For example, if we have another vector

$$\vec{w} = c\hat{x} + d\hat{y} \ ,$$

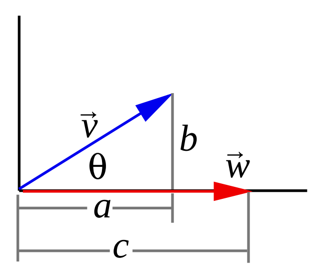
then

$$\vec{v} - 2\vec{w} = (a-2c)\hat{x} + (b-2d)\hat{y} \ .$$

Suppose we know the components of two vectors. Can we easily calculate their dot product? The answer is yes. The dot product of the above two vectors turns out to be just the sum of the products of their components:

$$\vec{v} \cdot \vec{w} = ac + bd \ .$$

To check this, consider the case where $\vec{w}$ points in the $\hat{x}$ direction. (If it doesn't, then convert everything to a new set of unit vectors in which it *does*.) This means we may set d=0.



Beginning with the definition of the dot product, we find

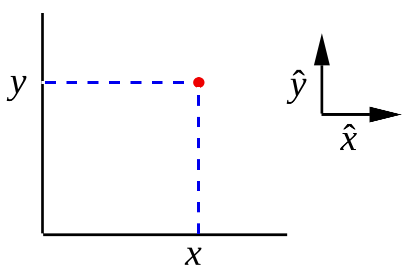
$$\begin{aligned}\vec{v}\cdot\vec{w} &= |\vec{v}|\,|\vec{w}|\,\cos\theta \\ &= \left(|\vec{v}|\cos\theta\right)c \\ &= ac\ ,\end{aligned}$$

as claimed.

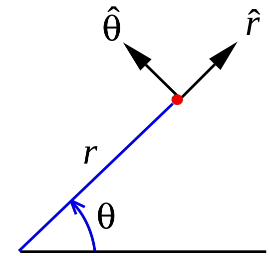
Polar coordinates and unit vectors

This section is included here mainly for future reference. You can safely skip it when you are reading about vectors for the first time.

The unit vectors we have just discussed are most appropriate when we are using *rectangular coordinates*. That is, we are specifying the location of any point in the plane by stating its x- and y-coordinates:

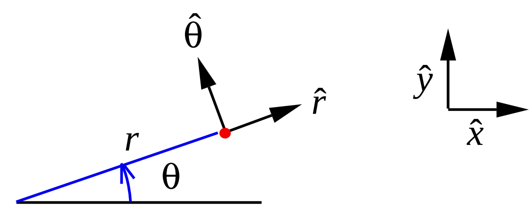


However, it is useful in many applications to specify points by their *polar coordinates*. These are r, the distance from the origin to the point, and θ, the angle measured clockwise from the x-axis:



In this case, the appropriate unit vectors to use are $\hat{r}$ and $\hat{\theta}$, as shown in the above figure. The former points in the direction of increasing radial coordinate r , while the latter points in the direction of increasing angle θ.

Note that these unit vectors are *not fixed*. Their direction depends on where they are. That is, they remain at right angles to one another, but both point in different directions depending on the value of θ. Compare the next figure with the previous one:



In contrast, the unit vectors $\hat{x}$ and $\hat{y}$ are *fixed*, once and for all.

For this reason, it is often useful to express $\hat{r}$ and $\hat{\theta}$ in terms of $\hat{x}$ and $\hat{y}$. It would be an excellent exercise for you to show that the relations are

$$\hat{r} = \hat{x}\cos\theta + \hat{y}\sin\theta$$

$$\hat{\theta} = -\hat{x}\sin\theta + \hat{y}\cos\theta \ .$$

Sometimes it's also useful to be able to go the other way. The inverse relations are

$$\hat{x} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

$$\hat{y} = \hat{r}\sin\theta + \hat{\theta}\cos\theta \ .$$

The velocity and acceleration vectors

Suppose we have a body which moves from place to place. Then its displacement vector will be a function of time:

$$\vec{r}(t) = \hat{x}x(t) + \hat{y}y(t) \ .$$

Don't let the notation confuse you; the things with the hats are just the usual fixed unit vectors, and *x(t)* and *y(t)* are the components of the displacement vector. They are functions of time.

The velocity vector is just the time derivative of the displacement vector:

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} \ .$$

The unit vectors we are using here don't depend on time, so the velocity vector in component form is

$$\vec{v}(t) = \hat{x}\frac{dx(t)}{dt} + \hat{y}\frac{dy(t)}{dt} \ .$$

Similarly, the acceleration vector is

$$\vec{a}(t) \equiv \frac{d\vec{v}(t)}{dt} = \hat{x}\frac{d^2x(t)}{dt^2} + \hat{y}\frac{d^2y(t)}{dt^2} \ .$$

Velocity and acceleration in polar coordinates

The expressions in the last section are given in rectangular coordinates. Sometimes, particularly when we are dealing with circular motion, it is useful to have expressions for position, velocity and acceleration in polar coordinates. The displacement vector is

$$\vec{r}(t) = \hat{r}\,r(t) \ .$$

Of course, this has no component in the θ-direction. In order to find the velocity, we have to differentiate and take into account the fact that $\hat{r}$ *changes direction* as the position of the particle changes:

$$\frac{d\vec{r}(t)}{dt} = \frac{d\hat{r}}{dt}\, r + \hat{r}\,\frac{dr}{dt} \quad .$$

Let's deal with this aspect first. Going back to our expression for $\hat{r}$ in terms of the fixed unit vectors $\hat{x}$ and $\hat{y}$, and differentiating, gives

$$\frac{d\hat{r}}{dt} = -\hat{x}\,\omega \sin\theta + \hat{y}\,\omega\cos\theta \quad .$$

We have used the common short-hand notation for the *angular velocity*

$$\omega = \frac{d\theta}{dt} \quad .$$

Notice that the right-hand side is just proportional to $\hat{\theta}$:

$$\frac{d\hat{r}}{dt} = \omega\,\hat{\theta} \quad .$$

This says that $\hat{r}$ just "twists around" in the $\hat{\theta}$-direction. Inserting into our earlier expression gives the final result

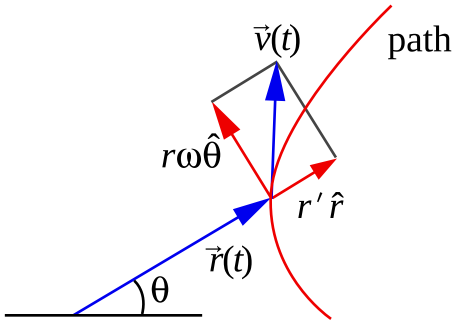
$$\boxed{\vec{v}(t) = r'\,\hat{r} + r\,\omega\,\hat{\theta}} \quad .$$

Here, we have used the shorthand notation

$${}' = \frac{d}{dt} \quad .$$

(Note: if you can't see the prime clearly, use Reader's magnification tool to increase the magification.)

The next figure shows a picture of the above equation. It gives the decomposition of the velocity into a *radial* part (the first term) and an *angular* part (the second term).



We differentiate again to find the acceleration:

$$\frac{d\vec{v}(t)}{dt} = r''\,\hat{r} + r'\frac{d\hat{r}}{dt} + (r'\omega + r\omega')\,\hat{\theta} + r\omega\frac{d\hat{\theta}}{dt} \quad .$$

In the same way as we worked out the time derivative of $\hat{r}$ before, we can show that

$$\frac{d\hat{\theta}}{dt} = -\omega\hat{r} \quad .$$

A little re-arranging then shows that the acceleration is

$$\boxed{\vec{a}(t) = \hat{r}\left(r'' - r\omega^2\right) + \hat{\theta}\frac{1}{r}\frac{d}{dt}\left(r^2\omega\right)} \quad .$$

For an application of these formulas, see the section on circular motion.

Vector fields

A *field* is anything which is defined at all points in some region of space. For example, the temperature of the air in a room has a value at each point in the room. The values could be different, or they could be the same.

In the case of temperature, the value of the field at any given point is a single number. Such a field is called a *scalar* field. Mathematically, a scalar field is specified by a single function of the coordinates, written $F(\vec{r})$. Notice that $\vec{r}$ is a vector (specifying the point in the room, for example), while the value $F(\vec{r})$ of the field at that point is a single number.

There are other kinds of fields besides scalar fields. The next most complicated kind is a *vector* field. As you might expect, that's a field whose value at each point is a vector. Such a thing is specified mathematically by as many functions as there are spatial dimensions. In two dimensions, it's written

$$\vec{F}(\vec{r}) = \hat{x}F_x(\vec{r}) + \hat{y}F_y(\vec{r}) \quad .$$

The *x*-component of the vector field is specified by a single function $F_x(\vec{r})$, while the *y*-component is specified by another function $F_y(\vec{r})$.

A *force field* is one type of vector field. Suppose the force on a body depends on where the body is located. Then the set of the force vectors at all points in space is a field. Examples include the gravitational field and the electric and magnetic fields, all of which we will study later.

There are other, higher kinds of fields. For example, a field whose value at each point is a *matrix* (an array of numbers) is called a *tensor* field. These fields are studied in later courses.

Time-dependent fields

Fields often depend on time. A time-dependent scalar field is written

$$F(\vec{r},t) \quad .$$

Here is a movie showing a time-dependent scalar field. The value of the field at a point in the horizontal plane is given by the vertical coordinate of the surface above that point.

A time-dependent vector field is written

$$\vec{F}(\vec{r},t) \ .$$

Here is a movie and another movie showing time-dependent vector fields.