Experiment 7

Angular Motion

Before starting the experiment be sure to review the descriptors of angular motion: angular position θ, angular velocity ω, angular acceleration α, torque τ, moment of inertia I.

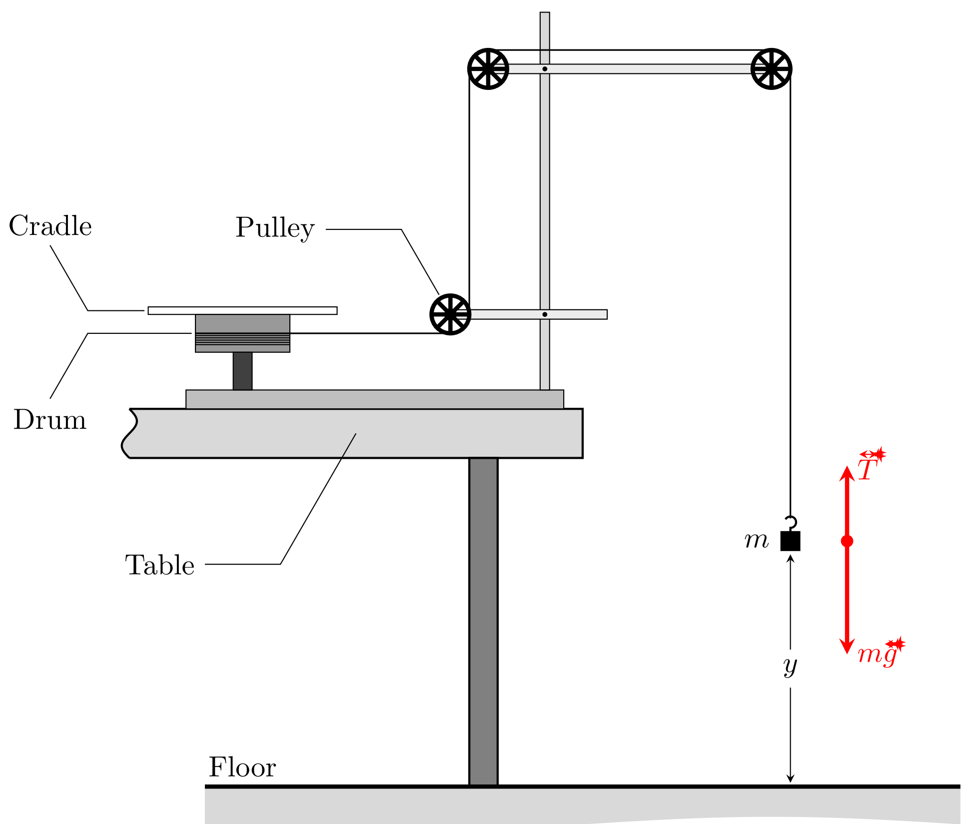

Figure 7.1: Experimental setup

The equipment in this Experiment consists of a plastic disc bolted to a metal drum (this combination is called the cradle), which can rotate nearly friction free, around a vertical axis, as illustrated in Figure 7.1. A torque τ can be applied to the cradle with a string-and-pulley combination and a pull mass m placed on a supporting holder. We arrange the pulleys so that the string leaving the drum is horizontal, *i.e.* perpendicular to the axis of rotation. The tension in the string is created by the weight of the pull mass at one end, and in turn exerts a torque on the cradle of magnitude

$$\tau = Tr \,, \tag{7.1}$$

where r is the radius of the pulley and T is the tension in the string.

If the string does not slip or stretch, the angular displacement of the cradle as a function of time $\theta(t)$ is related to the linear displacement of the pull mass $y(t)$, through $y(t) = r\theta(t)$. The downward linear acceleration is determined by the net force applied to the pull mass (Newton's second law for translations):

$$a = \frac{d^2y}{dt^2} = g - \frac{T}{m} \,. \tag{7.2}$$

Equivalently, the angular acceleration α is related to $a = r\alpha$, and in turn is caused by the applied torque (Newton's second law for rotations) through

$$\tau = I_c \, \alpha \,, \tag{7.3}$$

where I_c is the moment of inertia of the cradle.

7.1 Deriving an expression for rotational inertia

(!) Using Equations 7.1–7.3, show that for $g \gg a$, which is the case for all pull masses we will be using,

$$I_c = mr^2 \left(\frac{g}{a} - 1 \right) \approx mr^2 \frac{g}{a} \,. \tag{7.4}$$

7.2 Rotational inertia of an unloaded cradle assembly

(!) To acquire a data set $y(t)$ using `PhysTks`, click the [Hardware] menu and select the [Rotary encoder]. Set the rotary encoder output to represent a distance proportional to the pulley diameter used to rotate the cradle. All experiments will be completed with the string winding around the second smallest pulley on the cradle. Measured directly, the pulley radius is $r = 0.023$ m, but the translation of the angle measured by the rotary encoder into the distance travelled by the end of the string needs to be calibrated more precisely.

This is best done with a ruler. Take out the slack on the string, start data acquisition and rotate the platform by hand to raise the holder a distance of exactly 1.000 m above its starting position; 1 mm precision should be attained. The data in `PhysTks` should show a change from 0 m to 1.000 m. If not, the way to correct the calibration is to scale the value of r so that the 1.000-m distance reported in `PhysTks` corresponds to true 1 m. Adjust the r value and repeat the above as necessary.

Make sure that the centering overlay is mounted on the cradle, so that the moment of inertia of the cradle that you determine includes the contribution from it; this is required for Section 7.3. Place a mass on the holder and rotate the cradle to elevate the mass to a maximum height. Release the holder, then start the data collection. Terminate the data collection before the holder stops its free motion. If you miss that moment, trim the data set to remove the data points acquired when the motion of the holder has already stopped or was no longer free, and ensure that the string is still on the same pulley before the next attempt, as it may have slipped off the pulley.

We could fit the data to `A+B*x+0.5*C*x**2`, and the parameter `C` would represent the value of acceleration.

Perform at least two runs for each pull mass and calculate the averages. Your data record may look something like the one shown in Table 7.2, though you may have a different number of rows (one for

m, kg	a_1, m/s^2	a_2, m/s^2	$\langle a \rangle$, m/s^2	σ_a, m/s^2
0.010				
0.020				
0.030				
0.040				

Table 7.1: Example of a tabular record of acceleration data

each pull weight used) and columns (one for each repetition of the measurement). If you are planning to use `eXtrema` to do your data analysis (as is recommended), then only the first three columns of such a table need to be filled out, as the averages $\langle a \rangle$ and standard deviations σ_a should be calculated using `eXtrema` commands.

(!) To evaluate the rotational moment of inertia I_c of the unloaded cradle one can use a modified Equation 7.4, where we need to use the total mass $m_T = m + m_h - m_f$ in place of m. This total mass is what causes the tension T on the string, and in addition to the applied pull mass m, it includes the mass of the holder m_h and an effective mass m_f representing the static and dynamic friction on the platform and pulleys. You will have noted that the cradle sometimes begins to rotate with only the mass holder attached, when $m = 0$. In this case, $m_h > m_f$ and to prevent the system from rotating, m would need to be negative (or the mass holder would need to be lighter).

To get around the difficulty of not knowing m_f, replace m in Equation 7.4 with $m_T = m + m_h - m_f$ and express m as a function of a:

$$I_c = (m + m_h - m_f) r^2 \left(\frac{g}{a} - 1 \right) \approx (m + m_h - m_f) r^2 \left(\frac{g}{a} \right) = (m + m_h - m_f) \left(\frac{g r^2}{a} \right) , \tag{7.5}$$

and, therefore,

$$m = \left(\frac{I_c}{g r^2} \right) a + (m_f - m_h) . \tag{7.6}$$

The resulting equation is that of a straight line with slope $I_c/(g r^2)$ and the y-intercept $(m_f - m_h)$, and so knowing m_f and m_h directly is not necessary.

On a plot of m vs. $\langle a \rangle$, fit to a straight line to determine I_c.

7.3 Geometrical factors in front of mR^2: disc *vs.* ring

(!) Place the disc on the cradle (the centering overlay should already be on it, see Section 7.2) and repeat the measurements to obtain the total moment of inertia I_t of the cradle-plus-disc. Subtract I_c from I_t to obtain I_{disc}, the moment of inertia of the disc alone. Compare the measured value of I_{disc} with the theoretical value which you can calculate from the disc's mass and dimensions.

(!) Replace the disc with the ring on the cradle and repeat the measurements to obtain the total moment of inertia I_t of the cradle-plus-ring. Subtract I_c from I_t to obtain I_{ring}, the moment of inertia of the ring alone. Compare the measured value of I_{ring} with the theoretical value which you can calculate from the ring's mass and dimensions.

(!) The disk and ring are closely matched in both mass and diameter (they were made of different materials to achieve that). Calculate the ratio of the two moments of inertia, I_{disk}/I_{ring}. Is the value close to what you expected? Discuss the most likely reason for this discrepancy. For example, you may want to consider the effect of the ring having a finite thickness. The moment of inertia for such a ring can be calculated as $I_{ring} = \frac{1}{2}m(R_1^2 + R_2^2)$ where R_1 and R_2 are the inner and outer radii of the ring, the ring thickness being $(R_2 - R_1)$. In the limit of $R_1 \approx R_2 = R$, the familiar expression $I = mR^2$ is recovered.

7.4 Parallel axis' theorem

If we know I_0, the moment of inertia of a [possibly irregular] body of mass M about some axis, then the parallel axis theorem enables us to predict its moment of inertia I about any axis parallel to the original but displaced by a distance d:

$$I = I_0 + Md^2 \,. \tag{7.7}$$

In other words, under such a translation the body moves as a point mass, and we do not need to perform another integration over an irregular volume to calculate the moment of inertia. We will test this d-dependence by placing add-on masses at various distances from the center of the rotating cradle, and measuring the resulting dependence of the moment of inertia on this distance. A symmetric arrangement of two identical masses, as seen in Figure 7.2, is used to only change the moment of inertia, and not the position of the center of mass, which should prevent wobble and asymmetric forces on the cradle and rotary encoder. The slope of the graph plotted against d^2 should be exactly equal to M.

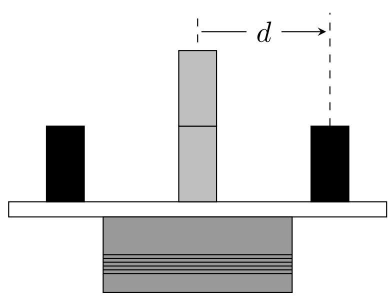

Figure 7.2: Adding off-center masses to the cradle in a symmetric manner (shown in black) changes the moment of inertia but not the position of the center of mass. Parallel axis theorem predicts an increase in the moment of inertia Md^2, compared to the original position of both masses in the center of the cradle (in light grey). M is the total mass of the two add-on masses placed symmetrically on the cradle.

Unfortunately, there is a significant friction in the apparatus, and it provides an additional force that is equivalent to a few grams of mass, which cannot be ignored if the pull masses are also only a few grams or tens of grams. Fortunately, we can modify our experimental procedure to separate the two contributions: the one from the force of the pull masses, and the one from friction in the pulleys/platform bearings/*etc.* Namely, we will start in the lowest position of the pull mass, and spin the platform up by hand (clockwise) so that its rotational inertia will lift up the pull mass first (the string will wind onto the pulley), reach a maximum height, turn around and come back down, now unwinding the string. The direction of the gravitational force on the pull mass remains the same, while the frictional force reverses direction because the direction of motion reverses.

Practice a few times before doing the data acquisition run: you want to be able to record a few seconds of upward motion, followed by a few seconds of downward motion. Do not give the platform too much of an initial spin, or the pull mass will fly beyond the upper pulley and possibly damage the apparatus. When you are confident of the amount of spin that is safe, and have a good estimate for the time it takes to

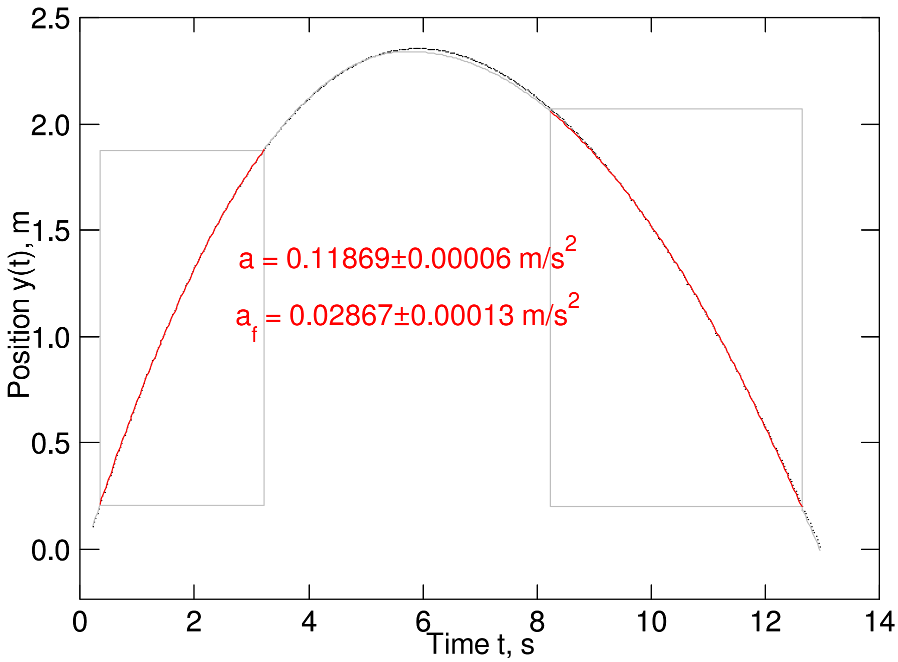

Figure 7.3: A typical trace $y(t)$ of the motion of the turntable. A gentle initial spin sends the holder with a pull mass moving up, slowed down by both the weight of the pull mass and the friction. After the peak of the trajectory, the direction of the gravitational pull remains the same, while the frictional force reverses its direction. The differences between the two sides of the trajectory can be analyzed to obtain the gravitational a and frictional a_f contributions to the acceleration. The grey boxes outline the sections of the $y(t)$ trace used to fit to a single piece-wise function of Equation 7.8. In this example, the section near the peak is ignored because at really slow speeds near the top, the string tends to get snagged between its own turns, and the data may not be reliable; this may not be necessary for your data.

complete a run, acquire a data set. Notice how the parabolic trajectory is lopsided, following one parabola on the way up, and a different one on the way down, as illustrated in a sample graph of Figure 7.3. You can select the appropriate regions of the graph, one at a time, and fit to `A+B*x+0.5*C*x**2`; the acceleration values will be different for the upward and downward portions of the motion.

Alternatively, you can import data into `eXtrema` and develop a macro to analyze the entire data set as one, by performing a fit to a piece-wise function

$$y(t) = y_0 - \begin{cases} \frac{1}{2}(a + a_f)(t - t_0)^2 \,, & \text{for } t < t_0 \\ \frac{1}{2}(a - a_f)(t - t_0)^2 \,, & \text{for } t \geq t_0 \end{cases} \tag{7.8}$$

where the peak position y_0 occurs at time t_0[1], and the acceleration due to friction a_f changes sign, while acceleration due to the force of gravity on the pull weight a maintains its direction. The parameters of the fit are y_0, t_0, a and a_f.

[1] At this point the velocity is momentarily zero, hence there is no need to include a term $v(t - t_0)$ in Equation 7.8.

Referring to Figure 7.3, the following code defines the rectangles that contain the data used to in the fit to determine the acceleration values for a_g and a_f:

```
t_block=5 ! t distance of fit rectangles from max y at t0
y_block=0.2 ! minimum y value of rectangles
!    fit command is a single line
fit\e2\weight (y>y_block)*((t<t0-1)+(t>t0+t_block))
y=y0-0.5*(a_g+a_f)*(t-t0)^2*(t<t0)-0.5*(a_g-a_f)*(t-t0)^2*(t>t0)
```

(!) Remove the centering overlay from the cradle to expose a series of holes drilled into the cradle along a radius. The holes have counterparts on the opposite side. The two cylindrical masses provided have pins on the bottom that should fit snugly into one of the holes, keeping the masses in place when the cradle begins to spin.

(!) Record the mass of both cylinders (you can treat both together as a single mass M) using a digital scale, then place them both directly in the center of the cradle. For this one measurement, the second mass is placed on top of the first one. Measure the moment of inertia as before; this is your data point for $d = 0$. Use at least four different pull masses, and fit a straight line to the m *vs.* a plot, as in Equation 7.6.

(!) Repeat, systematically moving the two cylindrical masses symmetrically outward on the cradle, for at least five or six distances from the center d. Be careful not to spin too fast when d gets large, as the masses may fly off the cradle and cause injury.

Determine M from the resulting d-dependence (see Equation 7.7), and compare it to the total mass of the two cylinders that you obtained directly using the scale.

The experimental apparatus you have used is relatively simple, but by designing the experimental procedure so as to avoid systematic errors caused by friction and other imperfections, very high-precision measurements could be made. In fact, the precision may be high enough that you may find that your result for M is not quite in perfect agreement with the weight scale value, the difference being greater than your experimental error. Discuss how the difference between the two measurements might be explained. Could it be possible that your error estimate is off? Consider also the influence of factors such as the mass of the string (its linear density is roughly 1 g/m); provide a qualitative argument or an order-of-magnitude calculation to determine whether a factor such as this could or could not account for the discrepancy.