Experiment 10

Coupled Oscillators

In this experiment you will examine the behaviour of coupled pendula, and investigate the dependence of the normal mode frequencies on the strength of the coupling. Before starting the experiment you have to be familiar with the concepts of normal modes, exchange and beat frequencies, and the theory of the simple physical pendulum.

10.1 Coupled oscillators

First, consider two oscillators made of two identical masses m and springs of identical spring constants k_0 attached to massive walls, as shown in Fig. 10.1. The positions of the two masses are given by the

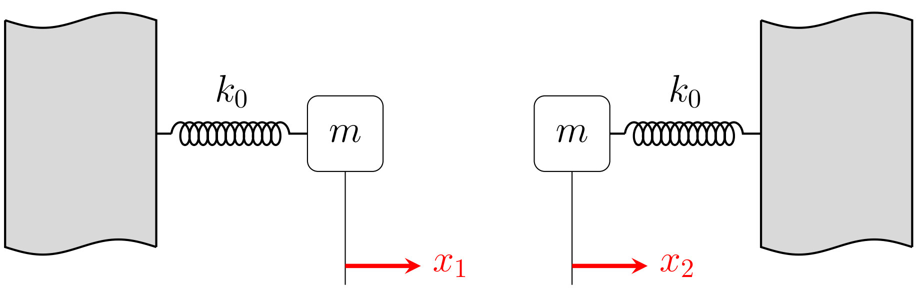

Figure 10.1: A schematic diagram of two separate, uncoupled oscillators.

coordinates $x_1(t)$ and $x_2(t)$; the x_1 and x_2 coordinates are measured from the neutral positions, the points where the springs are neither stretched nor compressed). The system is described by two uncoupled differential equations,

$$\ddot{x}_1 + \omega_0^2 x_1 = 0, \tag{10.1}$$
$$\ddot{x}_2 + \omega_0^2 x_2 = 0, \tag{10.2}$$

with the solutions:

$$x_1 = A_1 \cos(\omega_0 t + \psi_1), \tag{10.3}$$
$$x_2 = A_2 \cos(\omega_0 t + \psi_2), \tag{10.4}$$

with identical frequencies $\omega_0 = \sqrt{k_0/m}$ for the two oscillators, since the masses and spring constants are the same.

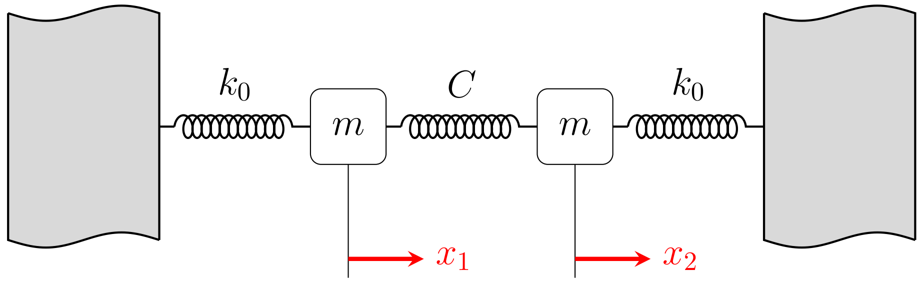

Figure 10.2: Adding a spring provides a coupling between oscillators.

Adding a spring with a spring constant C, to couple the two oscillators, as shown in Fig. 10.2, ensures that the motion of each oscillator directly influences the motion of the other, through the extensions and compressions of the coupling spring. The differential equations of motion acquire additional coupling terms:

$$\ddot{x}_1 + \omega_0^2 x_1 + k\,(x_1 - x_2) = 0, \tag{10.5}$$
$$\ddot{x}_2 + \omega_0^2 x_2 - k\,(x_1 - x_2) = 0, \tag{10.6}$$

where $k = C/m$.

Exercise #1

Derive Equations 10.5 and 10.6.

The solutions to Equations (10.5) and (10.6) are:

$$x_1 = \frac{1}{2}A_0\cos(\omega_0 t + \psi_0) + \frac{1}{2}A_1\cos(\omega_1 t + \psi_1), \tag{10.7}$$
$$x_2 = \frac{1}{2}A_0\cos(\omega_0 t + \psi_0) - \frac{1}{2}A_1\cos(\omega_1 t + \psi_1), \tag{10.8}$$

where $\omega_0^2 = k/m$ and $\omega_1^2 = \omega_0^2 + 2k$. The motion of the oscillators is now the superposition of *two* simple harmonic motions, with the (different) frequencies ω_0 and ω_1. The constants A_0, A_1, ψ_0 and ψ_1 depend on initial conditions.

The motions are particularly simple for the following cases:

1. If the initial conditions are

$$\left.\begin{array}{ll} \dot{x}_1 = 0, & \dot{x}_2 = 0 \\ x_1 = A, & x_2 = A \end{array}\right\} \text{at } t = 0\,,$$

then

$$x_1 = x_2 = A\cos\omega_0 t\,. \tag{10.9}$$

Both oscillators vibrate in phase at the same frequency ω_0; this motion is called the "first normal mode of vibration" for our system.

2. If the initial conditions are

$$\left.\begin{array}{ll} \dot{x}_1 = 0, & \dot{x}_2 = 0 \\ x_1 = A, & x_2 = -A \end{array}\right\} \text{at } t = 0\,,$$

then

$$x_1 = -x_2 = A\cos\omega_1 t\,. \tag{10.10}$$

This is the second normal mode.

3. If the initial conditions are

$$\left.\begin{array}{ll}\dot{x}_1 = 0, & \dot{x}_2 = 0\\ x_1 = 0, & x_2 = A\end{array}\right\}\text{at } t = 0\,,$$

then

$$x_1 = A\sin\frac{1}{2}(\omega_1-\omega_0)t\cdot\sin\frac{1}{2}(\omega_1+\omega_0)t, \tag{10.11}$$

$$x_2 = A\cos\frac{1}{2}(\omega_1-\omega_0)t\cdot\cos\frac{1}{2}(\omega_1+\omega_0)t. \tag{10.12}$$

Exercise #2

Derive Equations 10.11 and 10.12 from Equations 10.7 and 10.8 using the specified initial conditions.

Equation 10.11 says that the motion of the first oscillator is a harmonic motion with frequency $\frac{1}{2}(\omega_1 + \omega_0))$, but with an amplitude that itself changes in time as $A\sin\frac{1}{2}(\omega_1-\omega_0)t$. In other words, the amplitude itself is oscillating with a frequency $\frac{1}{2}(\omega_1-\omega_0)$. At time $t = 0$ the amplitude is zero; at $t = \pi/(\omega_1-\omega_0)$ the amplitude is A; at $t = 2\pi/(\omega_1-\omega_0)$ the amplitude is zero again and at $t = 3\pi/(\omega_1-\omega_0)$ the amplitude is $-A$, *etc.* A similar description applies to the motion of the second oscillator, described by Equation 10.12.

Exercise #3

Show that the amplitude of the second oscillator is equal to $\pm A$ when the amplitude of the first oscillator is zero, and vice versa.

The two oscillators periodically exchange their kinetic energy; the exchange period is $T_e = 2\pi/(\omega_1-\omega_0)$ and the exchange frequency $\omega_e = (\omega_1-\omega_0)$. That is: if at $t = 0$ the second oscillator has all the kinetic energy (and the first oscillator is at rest momentarily), then at time $t = \pi/(\omega_1-\omega_0)$ all the kinetic energy is in the first oscillator, (and the second oscillator is at rest). The "exchange period" is therefore: $T_e = 2\pi(\omega_1-\omega_0)$ and the exchange frequency: $\omega_e = (\omega_1-\omega_0)$.

10.2 Coupled pendula

First, let us consider a single pendulum, represented experimentally by a simple rod of length L and total mass M, suspended from and oscillating around one end. When the pendulum is given a small angular displacement α from the vertical, a torque J acts in it.

$$J = -Mg\,d\sin\alpha\,, \tag{10.13}$$

where g is the acceleration of gravity, and d is the distance from the centre of gravity to the axis of oscillation. The equation of motion of such a pendulum is

$$I\frac{d^2\alpha}{dt^2} = -Mg\,d\sin\alpha, \tag{10.14}$$

where I is the moment of inertia with respect to the axis of oscillation. For our pendulum, $I \cong \frac{1}{3}ML^2$ and $d \cong \frac{1}{2}L$. If the angle of oscillation α is small we may approximate $\sin\alpha \cong \alpha$, and Equation 10.2 becomes

$$\frac{d^2\alpha}{dt^2} + \omega_0^2\,\alpha = 0, \tag{10.15}$$

where $\omega_0^2 = Mgd/I$. We see from Equation 10.15 that the pendulum will execute harmonic oscillations with respect to the vertical with a frequency ω_0.

Now consider the situation when we couple the motions of two identical pendula by means of a spring with a spring constant K. Let the relaxed spring length be equal to the distance between the axes of oscillation, and let the spring be connected to the pendulum rods at a distance h from the axes of oscillation, as seen in Figure 10.3.

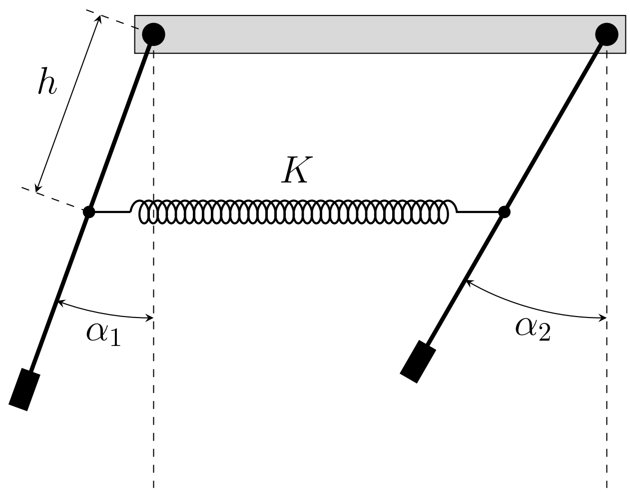

Figure 10.3: Two coupled pendula

The differential equation of motion of the first pendulum is:

$$I\frac{d^2\alpha_1}{dt^2} = -Mg\,d\sin\,\alpha_1 - K(h\sin\alpha_1 - h\sin\alpha_2)\cdot h\cos\,\alpha_1. \tag{10.16}$$

For small α this can be written as

$$\frac{d^2\alpha_1}{dt^2} = -\frac{Mgd}{I}\alpha_1 - \frac{Kh^2}{I}\alpha_1 + \frac{Kh^2}{I}\alpha_2. \tag{10.17}$$

Similarly, for the second pendulum the equation of motion is

$$\frac{d^2\alpha_2}{dt^2} = -\frac{Mgd}{I}\alpha_2 - \frac{Kh^2}{I}\alpha_2 + \frac{Kh^2}{I}\alpha_1. \tag{10.18}$$

Exercise #4

Derive Equation 10.17 by considering the torques on the first pendulum. *Hint:* torques are exerted on each pendulum by its weight and by the force of extended spring K.

Equations 10.17 and 10.18 are identical in form to Equations 10.5 and 10.6 of the two coupled oscillators, and those results can now be applied in a straightforward way to our system of two coupled pendula.

Exercise #5

Carefully compare Equations 10.5 and 10.6 with Equations 10.17 and 10.18. Find expressions for ω_0, ω_1 and k from the general theory in terms of I, K, M, d and h.

10.3 Experimental procedure

(!) Invoke the `PhysTks` data acquisition/plotting software and select `Rotary encoder` from the `Hardware` menu. Check that the USB cable from the pendulum apparatus is connected to your computer.

(!) Remove the coupling spring from two adjacent pendula P1 and P2 and move both spring holders to the $h = 100$ cm position at the extreme bottom of the pendulum rod.

(!) Acquire several seconds of P1 and P2 data, keeping α_1 and α_2 below 10°, then calculate their natural frequency ω_0 by selecting the desired data set with `Select #y` and fitting a sine wave to it. If necessary, tune the two pendula by adjusting the tuning screw at the bottom-end of one or both pendulum rods, so that the uncoupled pendula P1 and P2 have identical frequencies.

(!) Couple the two pendula with the spring at a distance $h = 100$ cm, then elevate and release one pendulum to set them swinging and acquire 15-20 seconds of data. To determine the frequencies ω_0 and ω_1 of the two normal modes and the exchange frequency ω_e using an Extrema script:

1. Plot the P1 and P2 data on separate graphs to view their $A(t)$ relationship.
2. Fit a sine wave to a plot of P1+P2 to get the normal mode frequency ω_0. You may need to include an exponential term to the fit equation if the data exhibits a noticeable decay.
3. Fit a sine wave to a plot of P1-P2 to get the normal mode frequency ω_1.
4. Calculate the exchange frequency $\omega_e = \omega_1 - \omega_0$.

(!) You can also determine ω_e by fitting Equation 10.11 to a pendulum data set.

1. Define two fit parameters, say $\omega_+ = (\omega_1 + \omega_0)/2$ and $\omega_- = (\omega_1 - \omega_0)/2$. Initial phase angles ϕ_0 and ϕ_1 will also be required by the fit.
2. Fit $y = A \sin(\omega_- t + \phi_0) \sin(\omega_+ t + \phi_1)$ to the P1 data. The exchange frequency is then $\omega_e = 2\omega_-$.

Compare this result with that obtained from a fit of the P2 data.

(!) Acquire several data sets using the same h but with different initial pendulum swing configurations, *i.e.* starting in phase, opposite phase or randomly as done above, then calculate average values $\langle\omega_1\rangle$ and $\langle\omega_e\rangle$.

(!) Decrease h in steps of 10 cm, to $h = 40$ cm, and for each h repeat the above procedure. Align the bottom of the spring holder with one of the index marks on the rod to set a distance h.

(!) Plot a graph of ω_1 as a function of h^2 and fit the points to a straight line.

The theoretical expectations are:

- the exchange frequency is $\omega_e = \omega_1 - \omega_0$; and
- the h-dependence of the oscillation frequency is $\omega_1 \propto h^2$.

Do your data obey these relations?