EM radiation (photons)

- Wave equation: $\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \nabla^2 \vec{E}$ | cf $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi$

subject to : $\operatorname{div}\vec{E} = \phi$ (no sources)

and BC : $E = \phi$ @ boundary

time dependence : $e^{-i\varepsilon_n t/\hbar} = e^{-i\omega_n t}$; $\omega_n = \frac{\varepsilon_n}{\hbar}$

ω_n values fixed by BC:

$$\omega_n^2 = c^2\left(\frac{\pi n}{L}\right)^2$$ | cf $\varepsilon_n = \frac{\hbar^2}{2m}\left(\frac{\pi n}{L}\right)^2$

$$\Rightarrow \boxed{\omega_n = \frac{c\pi n}{L}}$$

details see EM e.g. Jackson

- Rewrite: $n = \frac{L\omega_n}{\pi c}$ $\Rightarrow$ $\frac{dn}{d\omega} = \frac{L}{\pi c}$

$$\Rightarrow \mathcal{D}(\omega) = \frac{\pi n^2}{2}\gamma\frac{dn}{d\omega} = \frac{\pi}{2}\left(\frac{L\omega_n}{\pi c}\right)^2\gamma\frac{L}{\pi c} = \frac{\gamma}{2}\frac{\overset{=L^3}{V}}{\pi^2 c^3}\omega_n^2$$

Photons : $I=1$ so γ should be $=3$; but $\operatorname{div}\vec{E} = \phi$ requires that only 2 components of $\vec{E}$ are independent

$\Rightarrow \gamma = 2$. E.P. Wigner, Reviews of Modern Physics <u>29</u>: 255, 1957

$\Rightarrow$ Density of photon modes

$$\boxed{\mathcal{D}(\omega) = \frac{V}{\pi^2 c^3}\omega^2}$$

- occupation # of each mode is determined by the Boltzmann factor:

$$P(s) = \frac{1}{Z}e^{-\hbar\omega_n s/\tau}$$

! completely equivalent to a H.O. problem

$$\Rightarrow \langle s\rangle = \sum_{s=0}^{\infty} s\, e^{-\frac{\hbar\omega_n}{\tau}s} = \ldots \text{EFTS} \ldots = \frac{1}{e^{\hbar\omega_n/\tau}-1}$$

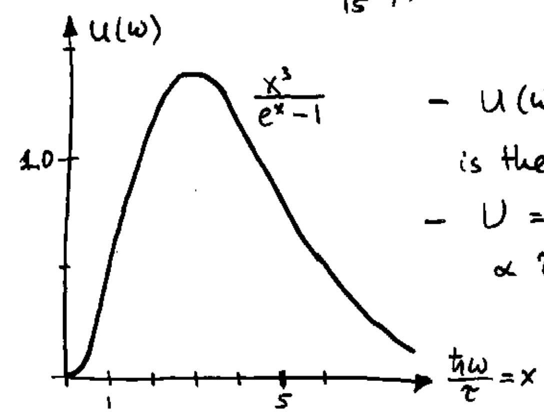
- $u(\omega_n) = \langle s\rangle \hbar\omega_n \mathcal{D}(\omega_n) = \frac{V\hbar}{\pi^2 c^3}\frac{\omega_n^3}{e^{\hbar\omega_n/\tau}-1}$

$$U = \sum_n u(\omega_n) \simeq \int_0^\infty d\omega\, u(\omega) = \frac{V\hbar}{\pi^2 c^3}\int_0^\infty d\omega \frac{\omega^3}{e^{\hbar\omega/\tau}-1}$$

$x = \frac{\hbar\omega}{\tau}$ $\Rightarrow$ $\omega^3 = \left(\frac{\tau}{\hbar}\right)^3 x^3$, $d\omega = \frac{\tau}{\hbar}dx$

$$\Rightarrow \boxed{U = \frac{V\hbar}{\pi^2 c^3}\left(\frac{\tau}{\hbar}\right)^4 \underbrace{\int_0^\infty dx \frac{x^3}{e^x-1}}_{=\frac{\pi^4}{15}\text{, from tables}} = \frac{\pi^2 V}{15\hbar^3 c^3}\tau^4}$$

Stefan-Boltzmann law of radiation

- 

- $u(\omega) \propto \frac{x^3}{e^x-1}$, $x = \frac{\hbar\omega}{\tau}$ is the Planck radiation law
- U = area under the curve $\propto \tau^4$ (Stefan-Boltzmann)

- max of energy : $\frac{d}{dx}\left(\frac{x^3}{e^x-1}\right) = \phi = \frac{3x^2}{e^x-1} - \frac{x^3 e^x}{(e^x-1)^2}$

$\Rightarrow 3e^x - 3 - xe^x = \phi$ $\Rightarrow$ $3 - 3e^{-x} = x$ solve numerically

$\Rightarrow x \simeq 2.82$ $\Rightarrow$ <u>$u(\omega)$</u> = max near $\frac{\hbar\omega}{\tau} = 2.82$

Ex. surface temperature of a star

$$\frac{\hbar\omega}{\tau} = \frac{\hbar\omega}{k_B T} = 2.82 \quad \Leftrightarrow \quad \lambda = \frac{2\pi\hbar c}{2.82\, k_B T} \simeq \frac{0.51}{T}\ \text{cm}$$

Sun: $u(\omega) = \max$ near $\lambda \simeq 0.8 \times 10^{-4}$ cm

$\Rightarrow T \approx 6000\,K$ @ Sun's surface

Ex cosmic background radiation peaks out @ $\lambda \simeq 0.16$ cm $\Rightarrow T \simeq 2.9\,K$ | one of the term projects

EFTS surface temperature of Earth due to re-radiation of Sun's energy KK Pr. 4-5

Elastic waves (phonons)

Very similar to photons, but:

- multiplicity $\gamma = 3$ (two transverse waves and one longitudinal wave possible)

  $\Rightarrow \mathcal{D}(n) = \frac{3}{2}\pi n^2$

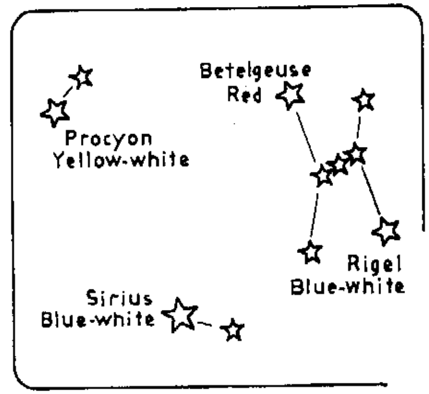
- assuming all modes propagate with the same velocity, $v = v_{sound}$, i.e. no dispersion

  $\Rightarrow \mathcal{D}(\omega) = \frac{3}{2}\frac{V}{\pi^2 v^3}\omega^2$

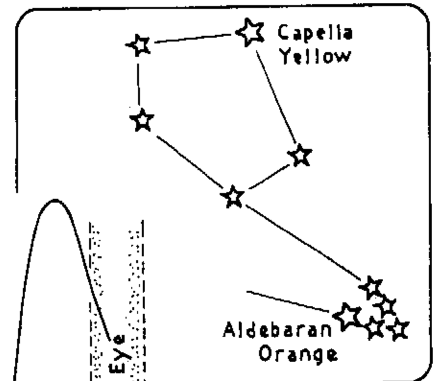
- the above only valid up to a maximum $\omega \equiv \omega_D$, after which $\mathcal{D}(\omega > \omega_D) = \phi$. $\omega_D = \omega_{Debye}$

"... every solid of finite dimensions contains a finite number of atoms and therefore has a finite number of free vibrations..."

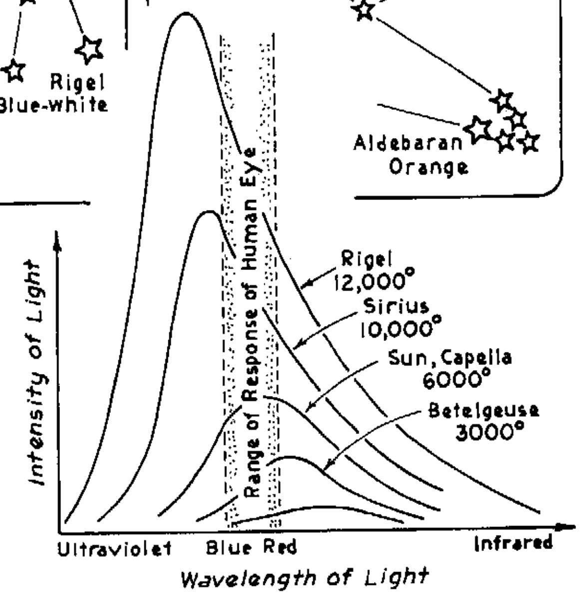
FEBRUARY 7



FEBRUARY 8



7th: Few people think that stars have color. Most of us recall the stars as diamondlike points of white light. But careful observation reveals a rich celestial palette. The winter sky offers an excellent opportunity to test your perception of the colors of the stars. The bluish cast of Sirius or Rigel is hard to miss. The ruddy glow of Betelgeuse and Aldebaran is also easy to distinguish. Capella, high overhead, is a yellow star like our sun, and appears so if you look carefully. Procyon, in the Little Dog, is a fierce yellow-white. The color of a star is determined by the temperature of the star's surface. The relationship is the same as for an iron poker in a fire. As the poker begins to heat up, it glows red hot, then orange, then yellow. If we continued to raise the temperature, the poker would appear white hot, or even white with a bluish cast. Match the color of the poker to the color of the star, and you have determined the temperature of the star.



8th: Light is an electromagnetic wave, and these waves can have differing wavelengths. Wavelength determines the color of light. Red light, for example, has a longer wavelength than blue light. Hot dense objects, like stars or pokers, emit a full rainbow of wavelengths called a *continuous spectrum*. The part of the spectrum to which the human eye is sensitive is called *visible light*. As the temperature of a luminous body increases, two things happen: the total brightness of the object increases, and the wavelength of peak intensity shifts toward the shorter wavelengths (or toward the blue end of the spectrum). It is the position in the spectrum of this peak intensity, relative to the visible part of the spectrum, that accounts for the color of the stars. The human eye is most sensitive to yellow light, possibly because we evolved near a yellow star!

Actual page from 365 Starry Nights

• find ω_D from normalization:

$$\int_0^\infty d\omega\, \mathcal{D}(\omega) \rightarrow \int_0^{\omega_D} d\omega\, \mathcal{D}(\omega) = 3N$$

since total # of elastic modes = # of degrees of freedom = $3N$

$$\Rightarrow \int_0^{\omega_D} d\omega\, \frac{3}{2}\frac{V}{\pi^2 v^3}\omega^2 = \frac{V}{2\pi^2 v^3}\omega_D^3$$

$$\Rightarrow \underline{\omega_D = \left(\frac{6\pi^2 N v^3}{V}\right)^{1/3}}$$

• $$U = \int_0^{\omega_D} d\omega\, \mathcal{D}(\omega)\langle s\rangle \hbar\omega = \frac{9N\hbar}{\omega_D^3}\int_0^{\omega_D} d\omega\, \frac{\omega^3}{e^{\hbar\omega/\tau}-1}$$

$x = \frac{\hbar\omega}{\tau} \Rightarrow \omega = \frac{\tau}{\hbar}x$ and $d\omega = \frac{\tau}{\hbar}dx$

$$\Rightarrow \boxed{U = \frac{9N\tau^4}{(\hbar\omega_D)^3}\int_0^{x_D} dx\, \frac{x^3}{e^x-1}}$$

where $x_D = \frac{\hbar\omega_D}{\tau} = \frac{\hbar\omega_D}{k_B}\frac{1}{T} = \frac{\Theta}{T}$

with $\Theta = \frac{\hbar\omega_D}{k_B}$ = Debye temperature

• In particular, at low temperatures: $T \ll \Theta$

$\Rightarrow x_D \gg 1 \Rightarrow$ approximate $\int_0^{x_D}$ by $\int_0^\infty$

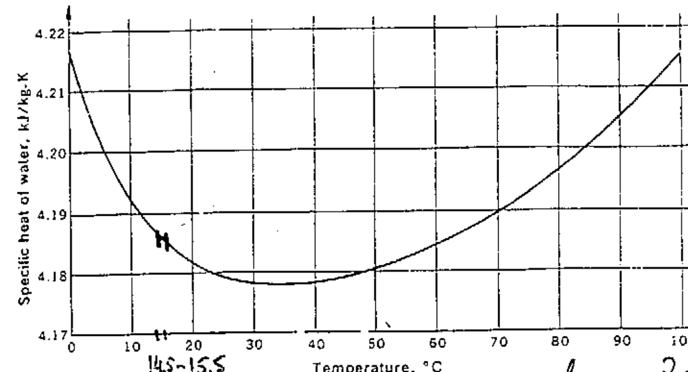
$$\Rightarrow U(T \ll \Theta) \simeq \frac{9N\tau^4}{(\hbar\omega_D)^3}\underbrace{\int_0^\infty dx\, \frac{x^3}{e^x-1}}_{=\frac{\pi^4}{15}} = \frac{3\pi^4 N k_B}{5\Theta^3}T^4$$

$$\Rightarrow \boxed{C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{12\pi^4 N k_B}{5\Theta^3}T^3 = \frac{12\pi^4 N k_B}{5}\left(\frac{T}{\Theta}\right)^3}$$

Debye T^3 law

see Fig. 4.10

EFTS examine $T \gg \Theta$ (Pr. 4-11)



H_2O a fluid

from Zemansky

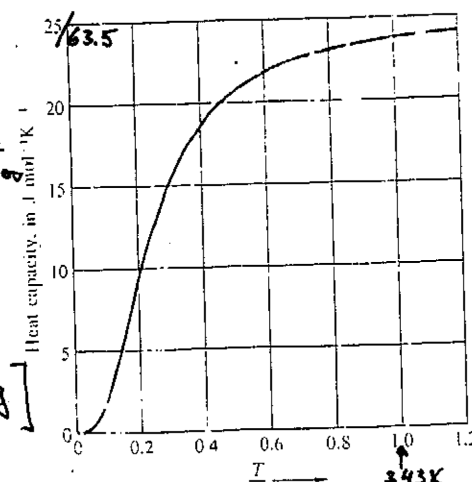


Figure 4.1. Heat capacity C_V of a solid according to the Debye approximation. The vertical scale is in J mol^{-1} K^{-1}. The horizontal scale is the temperature normalized to the Debye temperature θ. The region of the T^3 law is below 0.1θ. The asymptotic value at high values of T/θ is 24.943 J mol^{-1} K^{-1}.

[phonons only, ignoring C_V of electrons]

$\Theta_{Cu} = 343K$

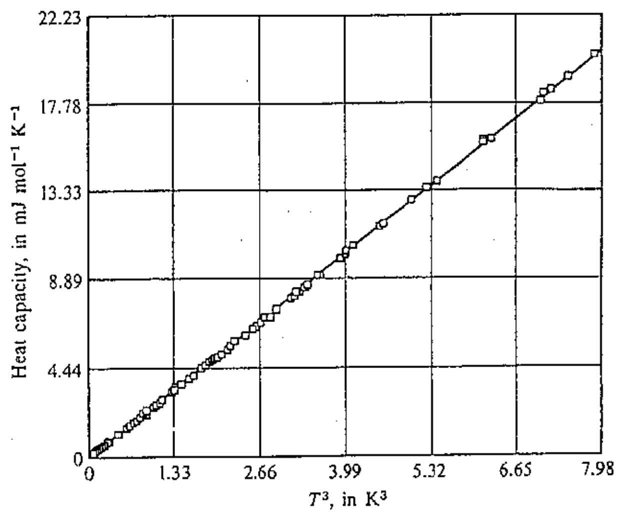


Figure 4.10 Low temperature heat capacity of solid argon, plotted against T^3 to show the excellent agreement with the Debye T^3 law. The value of θ from these data is 92 K. Courtesy of L. Finegold and N. E. Phillips.

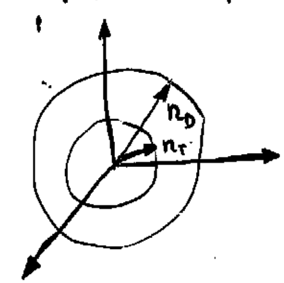
Table 4.1 Debye temperature θ_0 in K

Li 344	Be 1440											B	C 2230	N	O	F	Ne 75
Na 158	Mg 400											Al 428	Si 645	P	S	Cl	Ar 92
K 91	Ca 230	Sc 360	Ti 420	V 380	Cr 630	Mn 410	Fe 470	Co 445	Ni 450	Cu 343	Zn 327	Ga 320	Ge 374	As 282	Se 90	Br	Kr 72
Rb 56	Sr 147	Y 280	Zr 291	Nb 275	Mo 450	Tc	Ru 600	Rh 480	Pd 274	Ag 225	Cd 209	In 108	Sn w 200	Sb 211	Te 153	I	Xe 64
Cs 38	Ba 110	Laβ 142	Hf 252	Ta 240	W 400	Re 430	Os 500	Ir 420	Pt 240	Au 165	Hg 71.9	Tl 78.5	Pb 105	Bi 119	Po	At	Rn
Fr	Ra	Ac															
			Ce	Pr	Nd	Pm	Sm	Eu	Gd 200	Tb	Dy 210	Ho	Er	Tm	Yb 120	Lu 210	
			Th 163	Pa	U 207	Np	Pu	Am	Cm	Bk	Cf	Es	Fm	Md	No	Lr	

NOTE: The subscript zero on the θ denotes the low temperature limit of the experimental values.

Debye T^3 law: an intuitive argument

- traveling wave modes $e^{-i(\vec{n}\cdot\vec{r}-\omega t)}$
- no modes for $|\vec{n}| > n_D$
- at temperature $\tau = k_B T$, all modes below n_T have energy $\varepsilon = \tau = k_B T$ and above n_T (but below n_D) modes are not excited.
- total # of modes $= 3N$
- in n-space: 2 spheres



$$\frac{\text{total \# of excited modes}}{\text{total \# of modes} \approx 3N} = \left(\frac{n_T}{n_D}\right)^3 = \left(\frac{T}{\theta}\right)^3$$

$n \propto T$

- $U = k_B T \times 3N \times \left(\frac{T}{\theta}\right)^3 = 3N k_B \frac{T^4}{\theta^3}$

$\Rightarrow \quad C_V = 12 N k_B \left(\frac{T}{\theta}\right)^3$

We used $\langle n \rangle = \frac{1}{e^{\hbar\omega/\tau} - 1} \simeq \frac{1}{1 + \frac{\hbar\omega}{\tau} + \ldots - 1} \simeq \frac{\tau}{\hbar\omega}$

$\Rightarrow \quad \underline{\langle n \rangle \propto T}$