Green function of the Helmholtz Operator

1. In the Lorentz Gauge,

$$-\nabla^2 V + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = \frac{\rho}{\epsilon_o}$$

For a harmonically varying charge and current distribution

$$\rho(\vec{r},t) = \tilde{\rho}(\vec{r})e^{-i\omega t}$$

the above equation becomes:

$$-(\nabla^2 + k^2)V(\vec{r}) = \frac{\tilde{\rho}}{\epsilon_o}$$

The operator $-(\nabla^2+k^2)$ is called the Helmholtz operator. By assuming $|\vec{r} - \vec{r'}| = u$, that is position $\vec{r'}$ is at the origin of the $\vec{u}$ co-ordinate system, show that

$$\frac{1}{4\pi}\frac{e^{iku}}{u}$$

is the Green function of the Helmholtz operator. That is, prove that

$$-(\nabla^2 + k^2)\frac{1}{4\pi}\frac{e^{iku}}{u} = \delta^3(u)$$

which is the same as

$$-(\nabla^2 + k^2)\frac{1}{4\pi}\frac{e^{ik|\vec{r}-\vec{r'}|}}{|\vec{r}-\vec{r'}|} = \delta^3(\vec{r}-\vec{r'})$$

It is easiest to do this in spherical co-ordinates noting that ∇^2 acting on the co-ordinates of $\vec{r}$ assuming $\vec{r'}$ constant is the same as ∇^2 acting on the co-ordinates of $\vec{u}$.