Assignment No. 2

Physics 4P52

Due Monday, January 22, 2007

1. Obtain normalized common eigenvectors $|SM\rangle$ of $\hat{\mathbf{S}}^2 = (\hat{\mathbf{s}}_1 + \hat{\mathbf{s}}_2)^2$ and $\hat{S}_z = \hat{s}_{1z} + \hat{s}_{2z}$ for two spin-1/2 particles as follows (here we write $|+\rangle = |s = \frac{1}{2}\, m = \frac{1}{2}\rangle$ and $|-\rangle = |s = \frac{1}{2}\, m = -\frac{1}{2}\rangle$):

   (a) Start from $|S = 1\, M = 1\rangle = |s_1 = \frac{1}{2}\, m_1 = \frac{1}{2}\rangle|s_2 = \frac{1}{2}\, m_2 = \frac{1}{2}\rangle$ (this was shown in the class) and act on it twice with $\hat{S}_- = \hat{s}_{1-} + \hat{s}_{2-}$ to obtain normalized eigenvectors $|S = 1\, M = 0\rangle$ and $|S = 1\, M = -1\rangle$. (*Hint*: Apply $\hat{S}_-$ to the left-hand side and $\hat{s}_{1-} + \hat{s}_{2-}$ to the right-hand side using $\hat{K}_\pm|k\; m\; \lambda\rangle = \hbar\sqrt{k(k+1) - m(m \pm 1)}|k\; m - 1\; \lambda\rangle$ valid for general angular momentum.)

   (b) Find the linear combination of $|s_1 = \frac{1}{2}\, m_1 = \frac{1}{2}\rangle|s_2 = \frac{1}{2}\, m_2 = -\frac{1}{2}\rangle$ and $|s_1 = \frac{1}{2}\, m_1 = -\frac{1}{2}\rangle|s_2 = \frac{1}{2}\, m_2 = \frac{1}{2}\rangle$ that is normalized and orthogonal to $|S = 1\, M = 0\rangle$ obtained in (a). The resulting vector must be $|S = 0\, M = 0\rangle$. Explain why that is the case.

2. One can define the particle exchange operator $\hat{E}$ in the spin space $\mathcal{H}_s$ for two spin-1/2 particles as a linear operator such that (now we write $|+\rangle|+\rangle = |++\rangle$, $|+\rangle|-\rangle = |+-\rangle$, etc.)

$$\hat{E}|++\rangle = |++\rangle\,, \hat{E}|+-\rangle = |-+\rangle\,, \hat{E}|-+\rangle = |+-\rangle\,, \hat{E}|--\rangle = |--\rangle$$

   (i.e. it puts particle 1 into state in which particle 2 was, and vice versa).

   (a) Show that $\hat{E}$ is Hermitian and therefore that it is an observable.

   (b) Show that $\hat{E}^2 = \hat{1}$ (and therefore $\hat{E}^{-1} = \hat{E} = \hat{E}^\dagger$, so that $\hat{E}$ is also unitary).

   (c) Use (b) to show that the possible eigenvalues of $\hat{E}$ are +1 and -1. (The eigenvectors corresponding to the eigenvalue +1 are said to be symmetric under the exchange of particles, while the eigenvectors corresponding to the eigenvalue -1 are said to be antisymmetric under the particle exchange).

   (d) Show that spin singlet is antisymmetric, while the spin triplet states are symmetric under the exchange of particles in spin space.

Note: The exchange operator also acts on spatial states of particles, interchanging particles between two orbital states. Pauli principle states that the complete state (both orbital and spin) of several *identical* particles with half-integer spin must be antisymmetric under the exchange of *any* two particles. Hence for two identical spin-1/2 particles if the total spin state is singlet, the total orbital state must be symmetric under the particle exchange, while if the total spin state is a triplet state the total orbital state must be antisymmetric under the particle exchange.

3. (a) Show that

$$\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2 = \frac{1}{2}(\hat{\mathbf{S}}^2 - \hat{\mathbf{s}}_1^2 - \hat{\mathbf{s}}_2^2)$$

where, of course, $\hat{\mathbf{S}}^2 = (\hat{\mathbf{s}}_1 + \hat{\mathbf{s}}_2)^2$.
For two spins-1/2 $\hat{\mathbf{s}}_1^2 = 3\hbar^2/4 = \hat{\mathbf{s}}_2^2$ and then

$$\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2 = \frac{1}{2}\hat{\mathbf{S}}^2 - \frac{3}{4}\hbar^2 \, .$$

(b) Show that spin singlet and spin triplet states are the eigenvectors of $\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2$ and find the corresponding eigenvalues.

(c) If the interaction between two spins is of the form

$$\hat{H} = -J\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2$$

find the eigenvalues of $\hat{H}$ and the corresponding eigenvectors. If $J > 0$ (ferromagnetic coupling) show that the singlet state has higher energy than a triplet state and that therefore the system of two spins-1/2 will lower its energy by "aligning the spins".

(d) Use (a) to show that the operator $\hat{P}_1 = 3/4 + \hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2/\hbar^2$ annihilates the singlet state and multiplies a triplet state by 1. Thus acting on arbitrary two-spin-1/2 state $\hat{P}_1$ projects out the triplet part of the state.

(e) Show that $\hat{P}_0 = \hat{1} - \hat{P}_1 = 1/4 - \hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2/\hbar^2$ is a projection operator for the spin-singlet state.

4. Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and $\hat{A} = \hat{A}_1 \otimes \hat{A}_2 + \hat{B}_1 \otimes \hat{B}_2$ a correlated observable, where $\hat{A}_1$, $\hat{A}_2$, $\hat{B}_1$ and $\hat{B}_2$ are Hermitian operators in relevant factor spaces and $[\hat{A}_2, \hat{B}_2]=0$. The solution to the eigenvalue problem of $\hat{A}$ can be obtained in two steps:

- Find a solution to the common eigenvalue problem of $\hat{A}_2$ and $\hat{B}_2$ in $\mathcal{H}_2$

$$\begin{aligned} \hat{A}_2|a_2 b_2 \lambda\rangle &= a_2|a_2 b_2 \lambda\rangle \\ \hat{B}_2|a_2 b_2 \lambda\rangle &= b_2|a_2 b_2 \lambda\rangle \, . \end{aligned}$$

- Form an auxiliary operator $a_2\hat{A}_1+b_2\hat{B}_1$ in $\mathcal{H}_1$ and find a solution to its eigenvalue problem

$$(a_2\hat{A}_1 + b_2\hat{B}_1)|c\nu\rangle = c|c\nu\rangle$$

  (note that $c = c(a_2, b_2)$).

Then

$$\hat{A}(|c\nu\rangle \otimes |a_2b_2\lambda\rangle) = c(|c\nu\rangle \otimes |a_2b_2\lambda\rangle) \, .$$

(a) Prove the above theorem which is known as *the general theorem on separation of variables.*

(b) Apply this theorem to the eigenvalue problem of $\hat{\mathbf{l}}^2$ in $\mathcal{H}_\Omega = \mathcal{H}_\theta \otimes \mathcal{H}_\phi$ (see Assignment No. 1) since

$$\hat{\mathbf{l}}^2 = -\hbar^2 \left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right] = \frac{1}{\sin^2\theta}\otimes\hat{l}_z^2 + (-\hbar^2)\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)\otimes\hat{1}_\phi,$$

where $\hat{l}_z = -i\hbar d/d\phi$ in $\mathcal{H}_\phi$ and write down the eigenvalue problem of the auxiliary operator appearing in the second step (now $c = \hbar^2 l(l+1)$ as known from the general theory of angular momentum) as a differential equation.