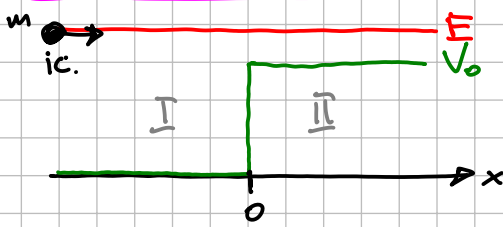


Q.M. wave propagation through a potential



classically: free particle will travel left-to-right, kinetic energy lower where $V_0 \neq 0$: $K = E - V_0$

Q.M. $\psi_I = A e^{i k_I x} + B e^{-i k_I x}$
 incident reflected

$$k_I^2 = \frac{E}{\hbar^2/2m}$$

$\psi_{II} = C e^{i k_{II} x} + D e^{-i k_{II} x}$
 transmitted

exclude by I.C.
 incident from II to I

$$k_{II}^2 = \frac{E - V_0}{\hbar^2/2m}$$

Match at the boundary, $x=0$:

$$\psi_I = \psi_{II} \rightarrow A + B = C$$

$$\psi'_I = \psi'_{II} \rightarrow k_I (A - B) = k_{II} C$$

} solve

$$\frac{C}{A} = \frac{2 k_I}{k_I + k_{II}}$$

$$\frac{B}{A} = \frac{k_I - k_{II}}{k_I + k_{II}}$$

current density $\vec{J} \equiv -\frac{i\hbar}{2m} (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*)$

so, for example, $J_c = C^* e^{-i k_{II} x} (i k_{II}) (C e^{i k_{II} x} - C e^{i k_{II} x} C^* (-i k_{II}) e^{-i k_{II} x})$
 $= 2 i k_{II} |C|^2$

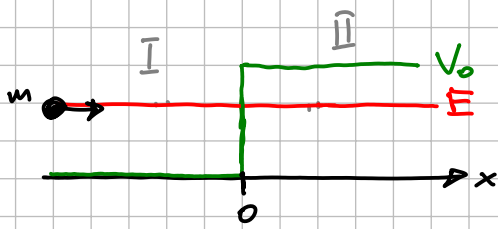
$\Rightarrow$ transmission & reflection coefficients are:

$$T \equiv \frac{J_c}{J_A} = \frac{k_{II} |C|^2}{k_I |A|^2} = \frac{k_{II} 4 k_I^2}{k_I (k_I + k_{II})^2} = \frac{4 k_I k_{II}}{(k_I + k_{II})^2}$$

$$R \equiv \frac{J_B}{J_A} = \frac{|B|^2}{|A|^2} = \left(\frac{k_I - k_{II}}{k_I + k_{II}} \right)^2$$

EFTS: check that
 $T + R = 1$

Note: R is the same for $k_I > k_{II}$ and $k_{II} > k_I$
 i.e. for step-up and step-down barriers



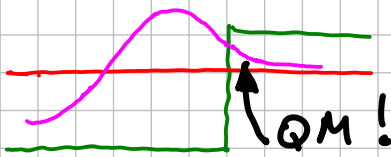
Classically, the particle is confined to I, i.e. it bounces back from the wall.

Q.M: same as before, but since $E - V_0 < 0 \Rightarrow k_{II} \rightarrow i k_{II}$

$$\Rightarrow \frac{C}{A} = \frac{2k_I}{k_I + i k_{II}} \quad \text{and} \quad \frac{B}{A} = \frac{k_I - i k_{II}}{k_I + i k_{II}}$$

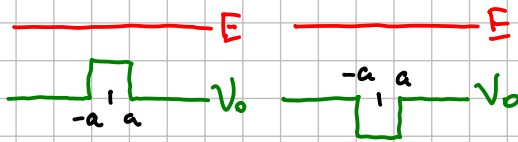
$$\Rightarrow R = \left(\frac{k_I - i k_{II}}{k_I + i k_{II}} \right)^2 = \frac{k_I^2 + k_{II}^2}{k_I^2 + k_{II}^2} = 1 \quad \text{i.e. also bounces back}$$

but this does not mean $\psi_{II} = 0$!



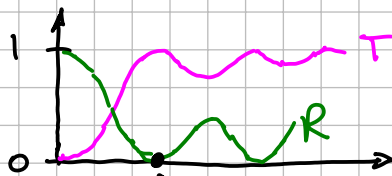
this apparent transparency of barriers is a purely Q.M. effect.

Ex:

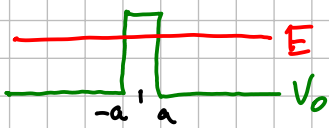


$E > V_0$: classically, a free particle

$$\text{Q.M. } \frac{1}{T} = 1 + \frac{1}{4} \frac{V_0^2}{E(E-V_0)} \sin^2 2k_{II}a$$



Ramsauer effect



Classically: an impenetrable barrier

Q.M:

$$\frac{1}{T} = 1 + \frac{1}{4} \frac{V_0^2}{E(V_0-E)} \sinh^2 2k_{II}a$$

QM tunnelling

• momentum wavefunctions

$$\begin{cases} \psi(x) = \frac{1}{\sqrt{2\pi}} \int \varphi(k) e^{ikx} dk \\ \varphi(k) = \frac{1}{\sqrt{2\pi}} \int \psi(x) e^{-ikx} dx \end{cases} \quad \begin{matrix} \leftarrow \text{coordinate} \\ \leftarrow \text{momentum} \end{matrix} \quad \begin{matrix} \text{representations} \\ \text{of the same} \\ \text{quantum state} \end{matrix}$$

$$\int |\psi(x)|^2 dx = \int |\varphi(k)|^2 dk = 1 \quad \begin{matrix} \leftarrow \text{normalization} \\ \text{(Parseval's theorem)} \end{matrix}$$

Interpretation: $|\varphi(k)|^2 dk$ = probability that momentum is between $\hbar k$ and $\hbar(k+dk)$

Ex a Gaussian wave "packet":

$$\varphi(k) = \frac{(2\Delta)^{1/2}}{(2\pi)^{1/4}} e^{-(k-k_0)^2 \Delta^2}$$

$$\Rightarrow \psi(x, 0) = (2\pi\Delta^2)^{-1/4} e^{-x^2/4\Delta^2 + ik_0 x}$$

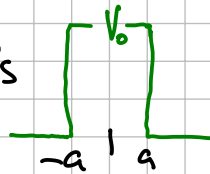
and

$$\psi(x, t) = (2\pi)^{-1/2} \int \varphi(k) e^{-i\omega t} e^{ikx} dk$$

$$= (2\pi)^{-1/4} \left(\Delta + \frac{it\hbar}{2m\Delta} \right)^{-1/2} \exp \left[\frac{-x^2/4\Delta^2 + ik_0 x - i\hbar \frac{k_0^2}{2m} t}{1 + i\hbar \frac{t}{2m\Delta^2}} \right]$$

EFTS1: verify this expression

EFTS2: introduce natural scales to make the above dimensionless, i.e. suitable for numeric calculations

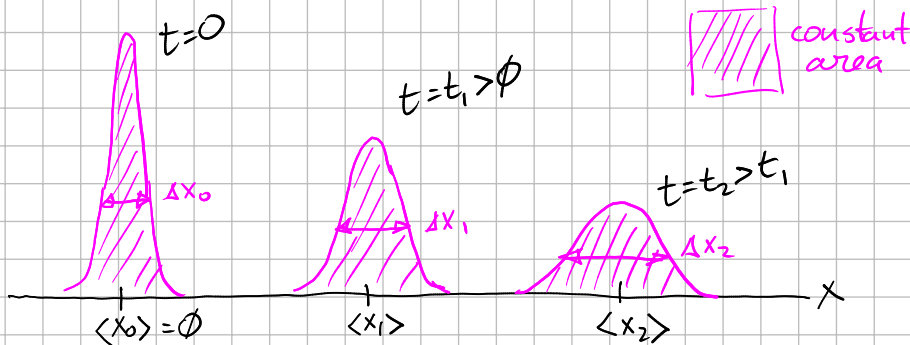
e.g. $x \rightarrow x' = \frac{x}{a}$ where V_0 barrier potential is 

Hint: need ε = energy scale so can choose, e.g.

$$V_0 = 16\varepsilon, \quad E = 4\varepsilon = \frac{1}{4}V_0 \text{ etc.}$$

one candidate: $\varepsilon = \frac{\hbar^2}{2m} \frac{1}{a^2}$ has dimensions of energy:

$$[\varepsilon] = \frac{(\text{Js})^2}{\text{kg m}^2} = \frac{\text{kg}^2 \text{m}^4}{\text{s}^2 \text{kg m}^2} = \frac{\text{kg m}^2}{\text{s}^2} = \text{J}$$



Qualitatively:

we have a range of p 's in the packet (i.e. a range of ψ 's) $\Rightarrow$ expect the wave packet to spread out as time goes on.