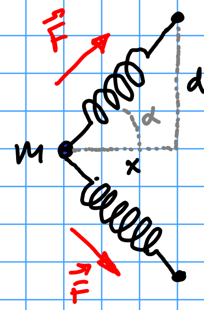
A note on anharmonicity



As in the lab, two springs (of natural length ℓ_0) connected as shown, will produce a force $F \propto x^3$

$$F = -\kappa\left(\sqrt{d^2+x^2} - \ell_0\right) \quad , \quad \cos\alpha = \frac{x}{\sqrt{d^2+x^2}}$$

$$\Rightarrow \quad m\ddot{x} = -2\kappa\left(\sqrt{d^2+x^2} - \ell_0\right)\frac{x}{\sqrt{d^2+x^2}}$$

$$m\ddot{x} = -2\kappa x + 2\kappa x \frac{\ell_0}{d}\frac{1}{\sqrt{1+(x/d)^2}}$$

Near the center, $\frac{x}{d} \ll 1 \Rightarrow (1+z)^{-1/2} \simeq 1 - \frac{1}{2}z + \dots$ for $z \ll 1$

$$\Rightarrow \quad m\ddot{x} = -\kappa x\left(2 - 2\frac{\ell_0}{d}\right) - \frac{\kappa x^3 \ell_0}{d^3}$$

In fact, for $d \approx \ell_0$, the 1st term vanishes and this is a purely anharmonic, $F \propto x^3$, oscillator. However, in general:

$$\Rightarrow \quad \boxed{\ddot{x} + \omega_0^2 x = -\beta x^3} \quad \text{with} \quad \boxed{\begin{array}{l} \omega_0^2 = \frac{\kappa}{m}\left(2 - 2\frac{\ell_0}{d}\right) \\ \beta = \frac{\kappa \ell_0}{m d^3} \end{array}}$$

Use perturbation theory approach (Landau)

$$x = x^{(1)} + x^{(2)} + \dots \quad \text{where} \quad x^{(1)} = A\cos\omega t$$

and $\omega = \omega_0 + \Delta\omega$ is the exact value (unknown).

Can re-write as

$$\underline{\frac{\omega_0^2}{\omega^2}\ddot{x}} + \ddot{x} + \omega_0^2 x = -\beta x^3 + \underline{\frac{\omega_0^2}{\omega^2}\ddot{x}}$$

here, we added the same extra term to both sides

$$\frac{\omega_0^2}{\omega^2}\ddot{x} + \omega_0^2 x = -\beta x^3 - \left(1 - \frac{\omega_0^2}{\omega^2}\right)\ddot{x}$$

But $x^{(1)} = A\cos\omega t \rightarrow \underline{\ddot{x}^{(1)} = -\omega^2 A\cos\omega t}$, $\Rightarrow \frac{\omega_0^2}{\omega^2}\ddot{x} + \omega_0^2 x = \emptyset$

With $x = x^{(1)} + x^{(2)}$, 1st order terms vanish on the L.H.side:

$$\Rightarrow \quad \underbrace{\frac{\omega_0^2}{\omega^2}\ddot{x}^{(2)} + \omega_0^2 x^{(2)}}_{\text{free } (\gamma=0) \text{ SHO @ true } \omega} = -\beta\left(x^{(1)}\right)^3 \underbrace{- \left(1 - \frac{\omega_0^2}{\omega^2}\right)\left(-\omega^2 A\cos\omega t\right)}_{\text{driven @ resonance } \omega}$$

$\Rightarrow$ runaway resonance! non-physical, i.e. the two terms on the R.H. side must cancel:

$$-\beta\left(x^{(1)}\right)^3 - \left(1 - \frac{\omega_0^2}{\omega^2}\right)\left(-\omega^2 A\cos\omega t\right) = \emptyset$$

$$-\beta A^3\cos^3\omega t + \left(1 - \frac{\omega_0^2}{\omega^2}\right)\omega^2 A\cos\omega t = \emptyset$$

Use: $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

$$\cos^3\omega t = \frac{3}{4}\cos\omega t + \frac{1}{4}\cos 3\omega t$$

$$\Rightarrow \quad -\beta A^3 \frac{3}{4}\cos\omega t + \left(1-\frac{\omega_0^2}{\omega^2}\right)\omega^2 A\cos\omega t + \underbrace{\ldots\cos 3\omega t}_{\text{far away from resonance}} = \emptyset$$

To avoid runaway resonance @ ω:

$$-\frac{3}{4}\beta A^2 + (\omega^2-\omega_0^2) = \emptyset$$

and since $\omega = \omega_0 + \Delta\omega$, $\omega^2 - \omega_0^2 = 2\omega_0\Delta\omega + \underbrace{\Delta\omega^2}_{\text{for small } \Delta\omega} \simeq 2\omega_0\Delta\omega$

$$\Rightarrow \quad \Delta\omega = \frac{3\beta A^2}{8\omega_0}$$

i.e. the observed resonance frequency will have a [weak] amplitude dependence:

$$\boxed{\omega = \omega_0 + \frac{3\beta}{8\omega_0}A^2}$$

Note the absence of the term linear in A.

This is only an approximation, as higher order terms – both in $x = x^{(1)} + x^{(2)} + \ldots$ and in the Fourier expansion, which we truncated to a single term $\cos(n\omega t)$, $n=1$, will contribute.

For a more complete description of $\omega(A)$ need a numerical solution.

Ref: a graduate course in Classical Mechanics

galileoandeinstein.physics.virginia.edu/7010/CM_22_Resonant_Nonlinear_Oscillations.html