Coriolis and Centrifugal Forces

We are now going to analyze a very simple physical situation from a less simple point of view.

Let's say we have a body which moves at constant velocity with respect to an inertial frame. No net force acts on it. Now suppose we look at this body from a frame of reference which is *accelerating*. In general, we no longer observe the body to be moving with constant velocity. It appears to accelerate, and from our point of view it looks as though there were a force acting on it. Such a force is called an "effective" or "fictitious" force. The acceleration due to such a force is caused solely by the motion of the observer.

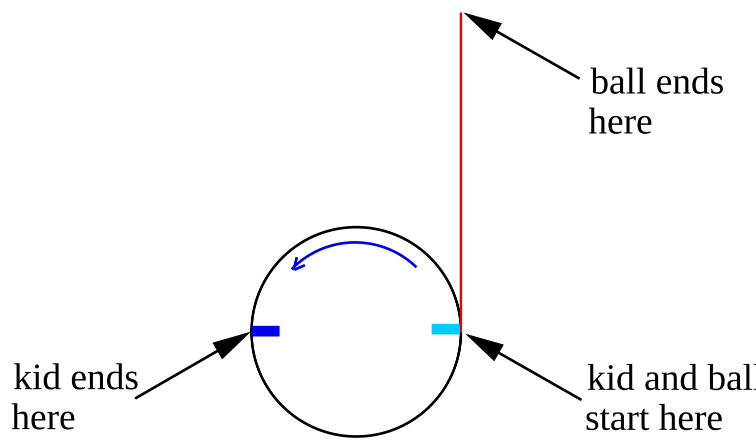
A common example of this effect is when you jump up and down, or spin around on your fancy desk chair. Everything appears to bounce up and down or go around and around, as the case may be.

In this section, we will be concerned with the particular case in which we are observing the uniformly-moving body from a frame of reference which is *rotating* at a constant rate. Our first example will involve a kid on a merry-go-round.

1. A kid on a merry-go-round

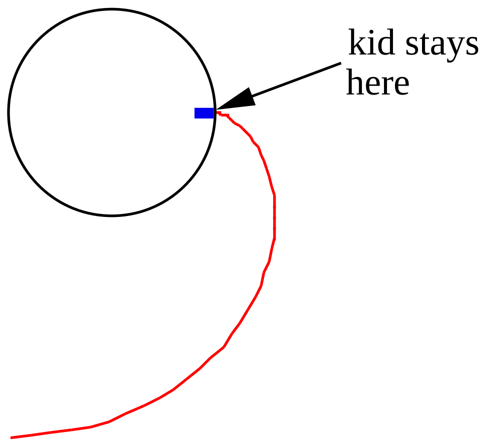
A kid is sitting on the edge of a merry-go-round, and he drops a ball. We will be interested in what happens in the horizontal plane - that is, when the whole thing is observed from above. Never mind the vertical motion of the ball.

If we are attached to the earth, viewing the whole thing from above, the ball just goes sailing off in a straight line as shown in this top view:



Here is a movie illustrating the ball's path.

The question is, what path does the ball follow, according to the kid? Here's the answer:



If you don't believe it, work it out! It's easiest to see if you realize that the kid has to keep turning his head, in order to see the ball. Here's a movie showing the motion seen by the kid.

According to the kid, there appears to be a force with one component pointing outwards and another directed around. How large are these components?

The calculation of the effective force seen by the kid is rather involved, and we won't attempt to reproduce it here. However, the result is remarkably simple, especially when written in vector notation. There are only two terms:

$$\vec{F}_{\text{eff}} = \underbrace{-m\vec{\omega}\times(\vec{\omega}\times\vec{r})}_{\text{centrifugal}} \underbrace{-2m\vec{\omega}\times\vec{v}}_{\text{coriolis}} \quad ,$$

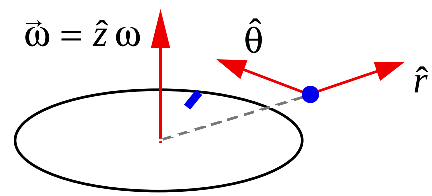
where $\vec{v}$ is the velocity as seen in the rotating frame (that is, as seen by the kid). The angular velocity vector of the rotating frame is $\vec{\omega}$. In our case, it points vertically out of the merry-go-round:

$$\vec{\omega} = \hat{z}\,\omega \quad .$$

Let's work out the centrifugal force. In plane polar coordinates, the body is located at

$$\vec{r} = \hat{r}\,r \quad .$$

The various vectors are shown in the next figure:



Using the relations

$$\hat{z}\times\hat{r} = \hat{\theta} \quad \text{and} \quad \hat{z}\times\hat{\theta} = -\hat{r} \; ,$$

we find

$$\boxed{\vec{F}_{\text{centrifugal}} = m\omega^2 r\,\hat{r}} \quad .$$

Hence, the centrifugal force is directed *outwards* from the center of rotation, and is exactly opposite to the centripetal force discussed earlier.

What about the coriolis force? If the velocity as seen in the rotating frame is

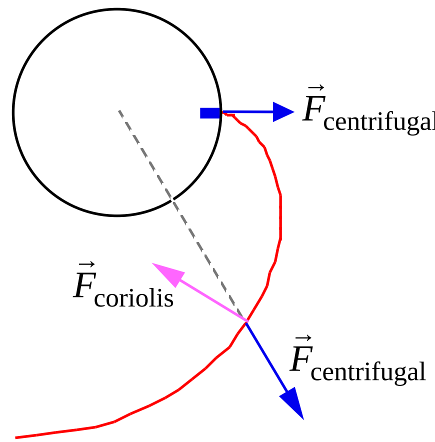
$$\vec{v} = \hat{r}\, v_r + \hat{\theta}\, v_\theta \,,$$

then the coriolis force is

$$\boxed{\vec{F}_{cor} = 2m\omega \left(\hat{r}\, v_\theta - \hat{\theta}\, v_r\right)} \,.$$

Notice that this has two parts, one in the radial direction and the other in the angular direction.

In our present example, at the instant the ball is released both components of the velocity are zero. The only apparent force is the centrifugal force, and this causes the radial component of the velocity to become nonzero. Therefore, the kid sees the ball come off the merry-go-round *at right angles* as seen by the kid. Then, since the radial component of the velocity is positive, the coriolis force points in the $-\hat{\theta}$ direction and the kid starts to see the ball go around.



Here is another movie of the motion observed by the kid, except this time the coriolis and centrifugal forces are shown.

Let's make an interesting and useful modification to the above situation. Suppose the kid throws the ball towards the center of the merry-go-round. What path will the kid observe the ball to follow?

If the kid defines "straight ahead" to mean "directly towards the center of the merry-go-round", then it turns out that he will see the ball *deflected to the right*. Here is a movie showing what the kid sees, and another showing the view from an observer fixed to the earth. The latter view makes it clear that the deflection observed by the kid is *entirely due to his rotational motion*.

2. Motion relative to the earth

We are now going to talk about the fictitious forces due to the rotation of the earth. This is what we were referring to in an earlier section, when we hinted that the surface of the earth is not really an inertial frame. You will see that the ideas of this section are just souped-up versions of the kid on the merry-go-round. All you have to do is look down from above the north pole.

Let's concentrate on the coriolis force. Suppose a body is located at (θ,ϕ) in spherical coordinates, and that its velocity *as seen on the earth* is given by

$$\vec{v} = \hat{r}\, v_r + \hat{\theta}\, v_\theta + \hat{\phi}\, v_\phi \quad .$$

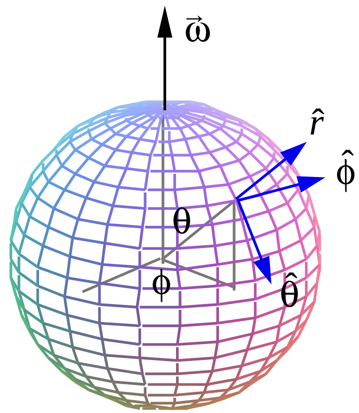
The earth's angular velocity vector is

$$\vec{\omega} = \hat{z}\, \omega \quad .$$

We have to work out the cross product

$$\vec{F}_{\text{coriolis}} = -2m\vec{\omega}\times\vec{v} \ ,$$

so we will need to know the cross products of the various unit vectors shown in the following figure:



Using the relations

$$\hat{z}\times\hat{r} = \hat{\phi}\sin\theta \quad , \quad \hat{z}\times\hat{\theta} = \hat{\phi}\cos\theta$$

and

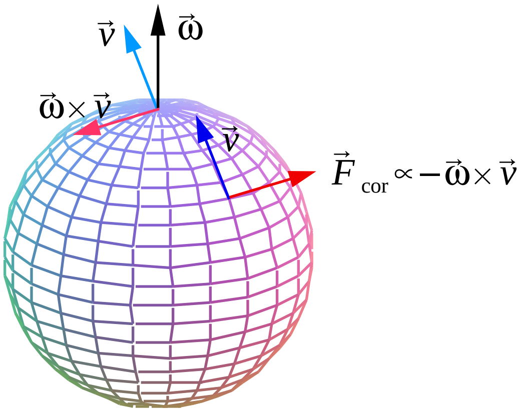
$$\hat{z}\times\hat{\phi} = -\hat{r}\sin\theta - \hat{\theta}\cos\theta \quad ,$$

it is easy to show that the coriolis force is

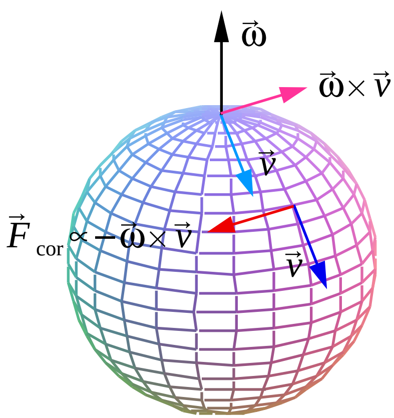
$$\vec{F}_{\text{cor}} = 2m\omega \left\{ \hat{r}\, v_\phi \sin\theta + \hat{\theta}\, v_\phi \cos\theta - \hat{\phi}\left(v_r \sin\theta + v_\theta \cos\theta \right) \right\} \quad .$$

The easiest way to make sense of this is to draw diagrams for some special cases:

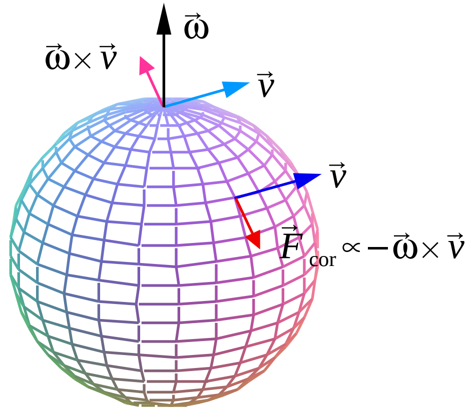
Body moving north (v_θ negative, other components zero); coriolis force eastward:



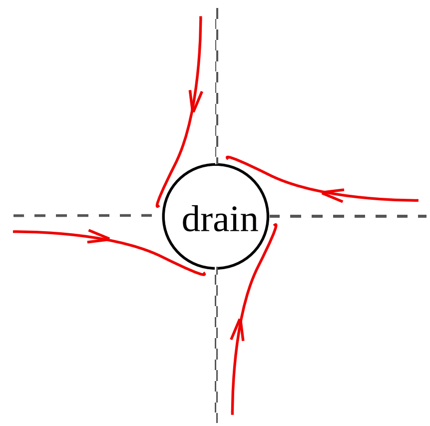
Body moving south (v_θ positive, other components zero); coriolis force westward:



Body moving east (v_ϕ positive, other components zero); coriolis force has component in southward direction:

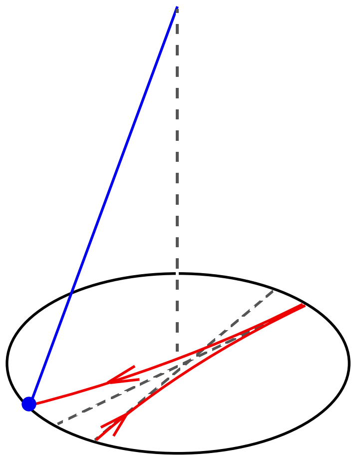


We see that for any body moving on the surface of the earth in the northern hemisphere, the coriolis force deflects it to the *right*. This is responsible for the counterclockwise rotation of the bath water as it drains from your tub, as viewed from above:



The same applies to the direction of air flow around an area of low atmospheric pressure.

Another case where the coriolis force is important is the *Foucault pendulum* (pronounced "Fooko"). This is a very large pendulum which you sometimes see in the lobbies of big important buildings. The plane in which the pendulum swings back and forth *precesses*, or turns, slowly around in a clockwise direction as viewed from above. The following diagram shows why:



As the pendulum swings across, the coriolis force pushes it to the right. On the way back it is also pushed to the right, and this just rotates the plane of the pendulum further in the clockwise sense. The Foucault pendulum is a rather striking demonstration of the rotation of the earth. Here is a movie showing the Foucault pendulum from above.

Last but not least, there is also an eastward force on a body falling vertically:

Body falling vertically (v_r negative, other components zero); coriolis force eastward: